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# An eight-dimensional realization of the Clifford algebra in the five-dimensional Galilean covariant spacetime

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## Abstract

We give an eight-dimensional realization of the Clifford algebra in the five-dimensional Galilean covariant spacetime by using a dimensional reduction from the  $(5 + 1)$  Minkowski spacetime to the  $(4 + 1)$  Minkowski spacetime which encompasses the Galilean covariant spacetime. A set of solutions of the Dirac-type equation in the five-dimensional Galilean covariant spacetime is obtained, based on the Pauli representation of  $8 \times 8$  gamma matrices. In order to find an explicit solution, we diagonalize the Klein–Gordon divisor by using the Galilean boost.

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## 1. Introduction

Nearly 20 years ago, Takahashi investigated the reduction from a  $(4 + 1)$  Galilean covariant manifold to the Newtonian spacetime (with three-dimensional space) in order to build the non-relativistic many-body theories by starting with Lorentz-like, manifestly covariant, equations [1]. This ‘Galilean covariant’ manifold is actually a  $(4 + 1)$  Minkowski spacetime with light-cone coordinates, which is reduced to the usual Newtonian spacetime [2]. Galilean covariant theories for the Dirac-type fields have been developed by using a four-dimensional realization of the Clifford algebra in a five-dimensional Galilean covariant spacetime [2]. Therein, we have 16 independent components that may be expressed as  $\gamma_A = \mathbb{I}, \gamma_\mu, \sigma_{\mu\nu}$ , with  $\mu, \nu = 1, \dots, 5$  [3]. Unfortunately, none of the pseudo-tensor interactions of ranks 0, 1 and 2 can be introduced into five- (or any odd-) dimensional theories, since they admit no ‘ $\gamma^6$  matrix’ which corresponds to the  $\gamma^5$  of the  $(3 + 1)$  Minkowski spacetime.

Motivated by the physical applications described in the following paragraphs, our purpose is to construct  $\gamma^5$ -like matrices in this Galilean covariant spacetime. A four-dimensional realization of the Clifford algebra in the  $(4 + 1)$  Minkowski spacetime requires  $\gamma^5$  as a fourth spatial element of  $\gamma_\mu$  s. Motivated by this fact, we discuss in this paper an eight-dimensional realization of the Clifford algebra in the  $(4 + 1)$  Galilean covariant spacetime. Thus our formulation involves two successive dimensional reductions: from the  $(5 + 1)$  Minkowski spacetime to a  $(4 + 1)$  Minkowski spacetime, which corresponds to the five-dimensional Galilean covariant spacetime mentioned earlier, and then from this extended manifold to the usual Newtonian spacetime [4].

Parity refers to a reversal of orientation of the spatial manifold. This corresponds to the reversal of coordinates in even-dimensional Minkowski spacetimes. In odd-dimensional spacetimes, in which the number of spatial coordinates is even, the reflection of spatial manifold has a determinant equal to one and hence it is continuously connected to the identity, and so can be obtained as a rotation. Therefore, we must define parity as the reversal of sign of an odd number of spatial coordinates in order to reverse the orientation of the spatial volume. This is the reason why we start in the  $(5 + 1)$  Minkowski spacetime in order to define a parity operation in the  $(4 + 1)$  Galilean covariant spacetime.

The development of eight-dimensional gamma matrices for the Dirac equation is motivated by applications to problems like the beta decay in the four-fermion Lagrangian of the  $V - A$  theory. This requires an evaluation of operators like

$$\bar{\psi}_h \gamma_\mu (1 - \gamma^5) \psi_h \bar{\psi}_l \gamma^\mu (1 - \gamma^5) \psi_l,$$

which are a combination of the hadron and lepton currents in Poincaré  $(3 + 1)$ -dimensional spacetime. Hence the necessity to have a  $\gamma^5$  matrix which provides us with a chirality operator. The leptonic part will be Poincaré invariant and the hadronic part will be Galilean invariant. The simplest example is the neutron decay:

$$n \longrightarrow p + e^- + \bar{\nu}.$$

This will provide us with an amplitude that still possesses a symmetry instead of just using an expansion in terms of  $p/m$ , thus destroying any symmetry in the hadronic part.

Another application of  $\gamma^5$  matrices is in deriving an  $N - N$  potential with a pseudo-vector or pseudo-scalar coupling. Although there is no Yukawa coupling in the Galilean covariant theories, it is still possible to define a four-point coupling. In addition, it is obvious that the interaction term has similarities with the Nambu–Jona–Lasinio theory [5]. Such a development may also be followed, in order to obtain further results, for the strongly interacting hadronic systems. Our purpose is to make progress along these lines with a Galilean covariant theory in  $(4 + 1)$  spacetime. However, in order to define the  $\gamma^5$ -like matrix, it is necessary to further extend the theories to a  $(5 + 1)$  Minkowski manifold. Results of this paper are therefore quite important in order to gain an understanding of the associated physical phenomena.

In section 2, we give an eight-dimensional realization of the Clifford algebra in the  $(5 + 1)$  Minkowski spacetime. Then, in section 3, we construct wavefunctions for the Dirac equation in this spacetime. By dimensional reduction from the  $(5 + 1)$  Minkowski spacetime to the  $(4 + 1)$  Minkowski spacetime, we obtain  $8 \times 8$  gamma matrices obeying the Clifford algebra in the  $(4 + 1)$  Galilean covariant spacetime in section 4. The construction of wavefunctions for the Dirac-type equation in the  $(4 + 1)$  Galilean covariant spacetime is performed in section 5. The final section contains concluding remarks.

We establish the commutation and anticommutation relations of  $8 \times 8$  gamma matrices in appendix A, and their trace formulae in appendix B. Fierz identities are developed in appendix C. Finally, in appendix D, we give explicit forms of wavefunctions obtained in sections 3 and 5. Throughout this work, we use the natural units, in which  $\hbar = 1$  and  $c = 1$ .

## 2. An eight-dimensional realization of the Clifford algebra in the (5 + 1) Minkowski spacetime

Let us consider a matrix representation of the complex Clifford algebra  $\overline{Cl}_6$  in a six-dimensional vector space with a Lorentzian signature (5 + 1). The complex Clifford algebra  $\overline{Cl}_6 \simeq Cl_{5,1} \otimes_{\mathbb{R}} \mathbb{C}$  is obtained by complexification of the real Clifford algebra  $Cl_{5,1}$ .

Let  $\gamma^\mu$  ( $\mu = 1, \dots, 5, 0$ ) be the  $8 \times 8$  Dirac gamma matrices. Then the  $\gamma$ -matrices satisfy the relation:

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}\mathbb{I}, \tag{1}$$

where we choose the metric tensor to be given by

$$g_{\mu\nu} = \text{diag}(1, 1, 1, 1, 1, -1) = g^{\mu\nu}, \tag{2}$$

such that

$$g_{\mu\lambda}g^{\lambda\nu} = \delta_\mu^\nu.$$

Also,  $\mathbb{I}$  in equation (1) denotes the  $8 \times 8$  unit matrix.

Some references about Clifford algebras are given in [6–8].

### 2.1. An eight-dimensional realization of the Clifford algebra

In order to obtain an explicit form of  $8 \times 8$  gamma matrices in a six-dimensional spacetime, and motivated by the fact that the basic building blocks of the matrix representation of  $\overline{Cl}_6$  are the Pauli matrices (defined below), we introduce the following nine matrices:

$$\begin{aligned} \rho &= \sigma \otimes I \otimes I, \\ \pi &= I \otimes \sigma \otimes I, \\ \Sigma &= I \otimes I \otimes \sigma, \end{aligned} \tag{3}$$

where  $\sigma$  are the Pauli matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then the matrices defined in equation (3) are

$$\begin{aligned} \rho_1 &= \begin{pmatrix} \mathbf{0}_{4 \times 4} & I & 0 \\ I & 0 & 0 \\ 0 & I & \mathbf{0}_{4 \times 4} \end{pmatrix}, & \rho_2 &= \begin{pmatrix} \mathbf{0}_{4 \times 4} & -iI & 0 \\ iI & 0 & -iI \\ 0 & iI & \mathbf{0}_{4 \times 4} \end{pmatrix}, & \rho_3 &= \begin{pmatrix} I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & I & 0 \\ \mathbf{0}_{4 \times 4} & -I & -I \end{pmatrix}, \\ \pi_1 &= \begin{pmatrix} 0 & I & \mathbf{0}_{4 \times 4} \\ I & 0 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & 0 & I \end{pmatrix}, & \pi_2 &= \begin{pmatrix} 0 & -iI & \mathbf{0}_{4 \times 4} \\ iI & 0 & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & 0 & -iI \end{pmatrix}, & \pi_3 &= \begin{pmatrix} I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & -I & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & I & -I \end{pmatrix}, \\ \Sigma &= \begin{pmatrix} \sigma & 0 & \mathbf{0}_{4 \times 4} \\ 0 & \sigma & \mathbf{0}_{4 \times 4} \\ \mathbf{0}_{4 \times 4} & \sigma & 0 \\ & 0 & \sigma \end{pmatrix}. \end{aligned}$$

The following relations hold among these matrices:

$$\begin{aligned} [\rho_k, \Sigma_l] &= [\pi_k, \Sigma_l] = [\rho_k, \pi_l] = 0, \\ \rho_k \rho_l &= \delta_{kl} + i\epsilon_{klm} \rho_m, \\ [\rho_k, \rho_l] &= 2i\epsilon_{klm} \rho_m, \\ \{\rho_k, \rho_l\} &= 2\delta_{kl}, \quad k, l, m = 1, 2, 3, \end{aligned}$$

with similar relations for  $\pi$ s and  $\Sigma$ s.

To complete our construction, we introduce three mutually orthogonal unit vectors,  $\mathbf{m}$ ,  $\mathbf{n}$ , and  $\mathbf{l} = \mathbf{m} \times \mathbf{n}$ , which are utilized to express the gamma matrices as follows:

$$\begin{aligned} \mathbf{m} \cdot \boldsymbol{\rho} &= i\gamma^0, \\ (\mathbf{m} \times \mathbf{n}) \cdot \boldsymbol{\rho} &= \gamma^7, \\ (\mathbf{n} \cdot \boldsymbol{\rho})(\mathbf{m} \cdot \boldsymbol{\pi}) &= \gamma^4, \\ (\mathbf{n} \cdot \boldsymbol{\rho})(\mathbf{l} \cdot \boldsymbol{\pi}) &= \gamma^5, \\ (\mathbf{n} \cdot \boldsymbol{\rho})(\mathbf{n} \cdot \boldsymbol{\pi}) \boldsymbol{\Sigma} &= \boldsymbol{\gamma}. \end{aligned} \quad (4)$$

We can prove that the  $\gamma^\mu$  s given by these equations satisfy the Clifford algebra (1), and that  $\gamma^7$  can be cast in the following form:

$$\gamma^7 = \frac{1}{6!} \epsilon_{\mu\nu\lambda\rho\sigma\tau} \gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\rho \gamma^\sigma \gamma^\tau = \gamma^1 \gamma^2 \gamma^3 \gamma^4 \gamma^5 \gamma^0,$$

with

$$\epsilon_{012345} = -\epsilon^{012345} = -\epsilon_{123450} = -1.$$

Let  $S$  denote the six-dimensional vector space corresponding to the eight-component Dirac spinor, and recall that the Lorentz generator may be written as

$$S^{\mu\nu} = \frac{1}{8} (\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu).$$

The matrices  $\gamma^\mu$  act on the representation  $S$  so that the representation of  $Cl_{5,1}$  is just  $\text{End}(S)$ , the set of endomorphisms of  $S$ . In a similar way, the matrices  $(\gamma^\mu)^T$  act on the dual representation  $S^*$ , the matrices  $(\gamma^\mu)^*$  act on the complex-conjugate representation  $\bar{S}$ , and the matrices  $(\gamma^\mu)^\dagger$  act on the representation  $\bar{S}^*$ , where the operations  $T$ ,  $*$  and  $\dagger$  denote the transpose, the complex conjugation and the hermitian conjugation operations, respectively. These representations are, in fact, equivalent because there exist elements of  $\text{End}(S)$  such that

$$(S^{\mu\nu})^\dagger = -\eta^{-1} S^{\mu\nu} \eta, \quad (S^{\mu\nu})^* = \hat{C}^{-1} S^{\mu\nu} \hat{C}, \quad (S^{\mu\nu})^T = -C^{-1} S^{\mu\nu} C.$$

The operators  $\eta$ ,  $\hat{C}$  and  $C$  are intertwining operators, e.g.  $\hat{C}$  intertwines the representations  $S$  and  $\bar{S}$ . Hence we can choose

$$\begin{aligned} (\gamma^\mu)^\dagger &= -\eta^{-1} \gamma^\mu \eta, \\ (\gamma^\mu)^* &= \hat{C}^{-1} \gamma^\mu \hat{C}, \\ (\gamma^\mu)^T &= -C^{-1} \gamma^\mu C. \end{aligned}$$

It is possible to choose  $\eta$  as

$$\eta \equiv i\gamma^0. \quad (5)$$

Then we find the following relations:

$$(\gamma^\mu)^T = -C^{-1} \gamma^\mu C = \hat{C}^{-1} (\gamma^\mu)^\dagger \hat{C}, \quad (6)$$

with

$$\hat{C} = \gamma^0 C, \quad (7)$$

and

$$\hat{C}^\dagger = \hat{C}^{-1} = -\hat{C}^*.$$

If a representation of  $\gamma^\mu$  (such as in equation (4)) is fixed, i.e. the vectors  $\mathbf{m}$ ,  $\mathbf{n}$  and  $\mathbf{l}$  are given, then the charge-conjugation matrix  $\hat{C}$  is determined by the relations (6) and (7), since  $\hat{C}$  is expressed in terms of the  $\gamma$ -matrices, defined in equation (4).

It is well known that the spinor representation,  $S$ , in the complex Clifford algebra  $\overline{Cl}_6$  is reducible, i.e.  $S = S_+ \oplus S_-$ . Indeed, the chirality matrix  $\gamma^7$ , defined earlier, satisfies

$$(\gamma^7)^2 = \mathbb{I}, \quad \{\gamma^\mu, \gamma^7\} = 0, \quad [\gamma^7, S^{\mu\nu}] = 0.$$

Hence,  $\gamma^7$  allows us to define the complex left- and right-handed Weyl spinors, which correspond to the two irreducible representations of  $\overline{Cl}_6$ :

$$\psi_\pm(x) = \frac{1}{2}(\mathbb{I} \pm \gamma^7)\psi(x),$$

with  $\psi_\pm(x) \in S_\pm$  and  $\psi(x) \in S$ .

Commutation and anticommutation relations involving the  $\gamma$ -matrices in a  $(5 + 1)$  Minkowski manifold are given in appendix A.1, and the corresponding trace relations are in appendix B.1.

In the next section, we discuss specific definitions of  $\mathbf{m}$ ,  $\mathbf{n}$  and  $\mathbf{l}$  which, in turn, lead to a particular representation of the Dirac matrices.

### 2.2. The Pauli representation of gamma matrices

In this section we construct a representation, in which  $i\gamma^0$  is diagonal, that we shall refer to as the ‘Pauli representation’ of  $\gamma$ -matrices. It is obtained by choosing

$$\mathbf{m} = (0, 0, 1), \quad \mathbf{n} = (0, 1, 0), \quad \mathbf{l} = (1, 0, 0).$$

Therefore, we find

$$\begin{aligned} i\gamma^0 &= \rho_3 = \sigma_3 \otimes I \otimes I = \begin{pmatrix} I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & I & \\ \mathbf{0}_{4 \times 4} & -I & 0 \\ & 0 & -I \end{pmatrix}, \\ \gamma^7 &= -\rho_1 = -\sigma_1 \otimes I \otimes I = \begin{pmatrix} & -I & 0 \\ \mathbf{0}_{4 \times 4} & 0 & -I \\ -I & 0 & \\ 0 & -I & \mathbf{0}_{4 \times 4} \end{pmatrix}, \\ \gamma^4 &= \rho_2\pi_3 = \sigma_2 \otimes \sigma_3 \otimes I = \begin{pmatrix} & -iI & 0 \\ \mathbf{0}_{4 \times 4} & 0 & iI \\ iI & 0 & \\ 0 & -iI & \mathbf{0}_{4 \times 4} \end{pmatrix}, \\ \gamma^5 &= \rho_2\pi_1 = \sigma_2 \otimes \sigma_1 \otimes I = \begin{pmatrix} & 0 & -iI \\ \mathbf{0}_{4 \times 4} & -iI & 0 \\ 0 & iI & \\ iI & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}, \\ \gamma &= \rho_2\pi_2\Sigma = \sigma_2 \otimes \sigma_2 \otimes \sigma = \begin{pmatrix} & 0 & -\sigma \\ \mathbf{0}_{4 \times 4} & \sigma & 0 \\ 0 & \sigma & \\ -\sigma & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}. \end{aligned} \tag{8}$$

Note that this representation is equivalent to the one described in [7, 8]. We can prove it by choosing the representation ( $\mathbf{m} = (1, 0, 0)$ ,  $\mathbf{n} = (0, 0, 1)$ ,  $\mathbf{l} = (0, 1, 0)$ ), which leads to

$$\begin{aligned} i\gamma^0 &= \rho_1 = \Sigma_1^{(3)}, \\ \gamma^7 &= -\rho_2 = -\Sigma_2^{(3)}, \\ \gamma^4 &= \rho_3\pi_1 = \Sigma_3^{(3)}, \\ \gamma^5 &= \rho_3\pi_2 = \Sigma_4^{(3)}, \\ \gamma^k &= \rho_3\pi_3\Sigma_k = \Sigma_{4+k}^{(3)}, \quad (k = 1, 2, 3), \end{aligned}$$

where  $\Sigma_a^{(3)}$  ( $a = 1, \dots, 7$ ) are in the notation defined in equation (4.1) of [7].

### 2.3. Number of independent gamma matrices

Let  $n$  be the dimension of spacetime, so that the number of  $\Gamma$ 's is  $2^n$ . Since we have

$$(1+x)^n = \sum_{k=0}^n \binom{n}{k} x^k = \sum_{k=0}^n {}_n C_k x^k,$$

and a completely antisymmetric tensor of rank  $k$  has  ${}_n C_k$  independent elements, then the number of independent  $\Gamma$ 's is

$${}_n C_0 + {}_n C_1 + \dots + {}_n C_n = \sum_{k=0}^n {}_n C_k (1)^k = (1+1)^n = 2^n.$$

Thus there exist  $2^n$  linearly independent matrices:

$$\Gamma_{\mu_1 \dots \mu_k}^{(k)} = d_{\mu_1 \dots \mu_k, \nu_1 \dots \nu_k}^{(k)} \gamma^{\nu_1} \dots \gamma^{\nu_k}, \quad (k = 0, 1, \dots, n),$$

where  $d^{(k)}$  are operators, described in [9], which project out the totally antisymmetric part of a rank- $k$  tensor.

In the case of six-dimensional Minkowski spacetime, we have  $2^6 = 64$  independent gamma matrices, which we write as

$$\begin{aligned} \Gamma^{(0)} &= \mathbb{I}, \\ \Gamma_{\mu}^{(1)} &= \gamma_{\mu}, \\ \Gamma_{\mu\nu}^{(2)} &= \sigma_{\mu\nu} = \frac{1}{2i}(\gamma_{\mu}\gamma_{\nu} - \gamma_{\nu}\gamma_{\mu}), \\ \Gamma_{\mu\nu\lambda}^{(3)} &= \sigma_{\mu\nu\lambda} = \frac{1}{3}(\gamma_{\mu}\sigma_{\nu\lambda} + \gamma_{\nu}\sigma_{\lambda\mu} + \gamma_{\lambda}\sigma_{\mu\nu}) = -\frac{1}{6}\epsilon_{\mu\nu\lambda\rho\sigma\tau}\sigma^{\rho\sigma\tau}\gamma^7, \\ \Gamma_{\mu\nu\rho\sigma}^{(4)} &= -\frac{1}{2}\epsilon_{\mu\nu\rho\sigma\xi\eta}\sigma^{\xi\eta}\gamma^7, \\ \Gamma_{\mu\nu\rho\sigma\lambda}^{(5)} &= -\epsilon_{\mu\nu\rho\sigma\lambda\xi}\gamma^{\xi}\gamma^7, \\ \Gamma_{\mu\nu\rho\sigma\lambda\tau}^{(6)} &= -\epsilon_{\mu\nu\rho\sigma\lambda\tau}\gamma^7. \end{aligned}$$

To show properties under the Lorentz transformations, we choose the following 64 linearly independent matrices:

$$\gamma_A = \mathbb{I}, \gamma^7, \gamma_{\mu}, i\gamma^7\gamma_{\mu}, \sigma_{\mu\nu}, \gamma^7\sigma_{\mu\nu}, \sigma_{\mu\nu\lambda},$$

satisfying

$$\begin{aligned} \gamma^A \gamma_A &= \mathbb{I}, & (\text{no summation over } A), \\ \text{Tr}(\gamma_A) &= 0, & \text{if } \gamma_A \neq \mathbb{I}, \end{aligned}$$

as well as

$$\text{Tr}(\gamma^A \gamma_B) = 8\delta_B^A.$$

By using the charge-conjugation matrix  $C$  of equation (6), we can separate the  $\gamma_A$  s into symmetric and antisymmetric elements as

$$(\gamma_A C)_{\alpha\beta} = \epsilon_A (\gamma_A C)_{\beta\alpha}, \quad (9)$$

or, equivalently,

$$(\gamma_A)_\alpha^\beta = \epsilon_A (C^{-1} \gamma_A C)_\alpha^\beta,$$

where

$$\epsilon_A = \begin{cases} +1 & \text{for } C, \gamma^7 \sigma_{\mu\nu} C, \sigma_{\lambda\mu\nu} C, \\ -1 & \text{for } \gamma^7 C, \gamma_\mu C, i\gamma^7 \gamma_\mu C, \sigma_{\mu\nu} C, \end{cases} \quad (10)$$

We have used the relation  $C^\dagger = C^{-1} = C^*$ .

Note that the lowercase indices from the beginning of the Greek alphabet,  $\alpha, \beta, \gamma$ , etc denote spinor indices, and the lowercase indices from the middle of the alphabet,  $\xi, \kappa, \lambda$ , etc are tensor indices.

#### 2.4. Parity

The parity matrix, denoted by  $\Pi$ , is defined by imposing the condition that the equation of motion be invariant under the discrete transformation of space reflection:

$$x^\mu \rightarrow x'^\mu = (-\mathbf{x}, x^4, x^5, x^0).$$

Consider the Dirac field, then the requirement reads

$$\eta^{-1} \Pi^\dagger \eta \gamma^\mu \Pi = \begin{cases} -\gamma^\mu, & \text{for } \mu = 1, 2, 3, \\ \gamma^\mu, & \text{for } \mu = 4, 5, 0, \end{cases} \quad (11)$$

where the matrix  $\eta$  is defined in equation (5). Hence the Dirac equation is invariant under the space reflection.

The parity matrix may be expressed by

$$\Pi = \gamma^4 \gamma^5 \gamma^0.$$

### 3. Construction of wavefunctions for the Dirac equation in the (5 + 1) Minkowski spacetime

In this section, we obtain the wavefunctions for the Dirac equation of motion in the extended (5 + 1) Minkowski manifold. We adopt the methods of constructing wavefunctions developed by Takahashi [10], in which the Klein–Gordon divisor is diagonalized by using the Lorentz boost.

The Dirac equation for massive particles with mass  $m$  is expressed in the form:

$$\Lambda(\partial)\psi(x) = 0, \quad (12)$$

where the operator  $\Lambda(\partial)$  is given by

$$\Lambda(\partial) = -(\gamma \cdot \partial + m).$$

Here, the scalar product is denoted by  $A \cdot B$  and defined by

$$A \cdot B = g_{\mu\nu} A^\mu B^\nu = A^i B^i + A^a B^a - A^0 B^0,$$



where the lowercase indices from the beginning of the Latin alphabet,  $a, b, c$ , etc take the values 4 and 5, and the lowercase indices from the middle of the Latin alphabet,  $i, j, k$ , etc run from 1 to 3.

The adjoint equation to equation (12) is obtained by taking its Hermitian conjugate:

$$\overline{\psi}(x)\Lambda(-\overleftarrow{\partial}) = 0,$$

( $\overleftarrow{\partial}$  denotes the left-derivative) with

$$\overline{\psi}(x) = \psi^\dagger(x)\eta.$$

We assume the existence of a non-singular matrix  $\eta$  which satisfies the relation:

$$[\eta\Lambda(\partial)]^\dagger = \eta\Lambda(-\partial). \quad (13)$$

This condition is equivalent to requiring the hermiticity of the Lagrangian in the form

$$\mathcal{L}(x) = \overline{\psi}(x)\Lambda(\partial)\psi(x).$$

Thus we choose  $\eta$  as defined in equation (5).

The operator  $d(\partial)$ , reciprocal to the operator  $\Lambda(\partial)$  of equation (12), is defined by

$$\Lambda(\partial)d(\partial) = d(\partial)\Lambda(\partial) = (\partial^2 - m^2)\mathbb{I}.$$

This reciprocal operator is called the ‘Klein–Gordon divisor’. It is given by

$$d(\partial) = -(\gamma \cdot \partial - m).$$

The Dirac field  $\psi(x)$  and its charge-conjugate field  $\psi_C(x)$  can be expanded in terms of  $c$ -number wavefunctions with positive and negative frequencies, represented by  $u_p^{(r)}(x)$  and  $v_p^{(r)}(x)$ , respectively, and two kinds of creation and annihilation operators:

$$\begin{aligned} \psi(x) &= \sum_r \int d\mathbf{p} d^2 p^a [u_p^{(r)}(x)a^{(r)}(\mathbf{p}, p^a) + v_p^{(r)}(x)b^{(r)\dagger}(\mathbf{p}, p^a)], \\ \psi_C(x) &:= \hat{C}\psi^*(x), \\ &= \sum_r \int d\mathbf{p} d^2 p^a [u_p^{(r)}(x)b^{(r)}(\mathbf{p}, p^a) + v_p^{(r)}(x)a^{(r)\dagger}(\mathbf{p}, p^a)], \end{aligned}$$

where

$$\begin{aligned} \{a^{(r)}(\mathbf{p}, p^a), a^{(r')\dagger}(\mathbf{p}', p'^a)\} &= \delta_{rr'}\delta(\mathbf{p} - \mathbf{p}')\delta^{(2)}(p^a - p'^a), \\ \{b^{(r)}(\mathbf{p}, p^a), b^{(r')\dagger}(\mathbf{p}', p'^a)\} &= \delta_{rr'}\delta(\mathbf{p} - \mathbf{p}')\delta^{(2)}(p^a - p'^a), \end{aligned}$$

and all other commutators of similar type vanish. We use the notation

$$d^2 p^a = d p^4 d p^5 \quad \text{and} \quad \delta^{(2)}(p^a - p'^a) = \delta(p^4 - p'^4)\delta(p^5 - p'^5).$$

The function  $v_p^{(r)}(x)$  is defined by

$$v_p^{(r)}(x) = \hat{C}u_p^{(r)*}(x).$$

The charge-conjugation matrix  $\hat{C}$ , defined by equation (6), satisfies

$$[\eta\Lambda(\partial)]^T = [\eta\Lambda(-\partial)]^* = -\hat{C}^{-1}\eta\Lambda(-\partial)\hat{C}.$$

It is convenient to take the functions  $u_p^{(r)}(x)$  to be eigenvectors of the operator  $-i\partial_\mu$ :

$$-i\partial_\mu u_p^{(r)}(x) = p_\mu u_p^{(r)}(x).$$

By substituting the Fourier transform of  $u_p^{(r)}(x)$  into this equation, we find

$$u_p^{(r)}(x) = f_p(x)u^{(r)}(\mathbf{p}, p^a), \quad v_p^{(r)}(x) = f_p^*(x)v^{(r)}(\mathbf{p}, p^a),$$

where

$$f_p(x) = (2\pi)^{-5/2} e^{ip \cdot x},$$

and

$$p^0 = \sqrt{\mathbf{p} \cdot \mathbf{p} + (p^4)^2 + (p^5)^2 + m^2}.$$

By following the prescription developed in chapter 5 of [10], we obtain the orthonormality condition and the closure properties in the momentum representation:

$$\begin{aligned} \bar{u}^{(r')}(\mathbf{p}, p^a) i\gamma^0 u^{(r)}(\mathbf{p}, p^a) &= \delta_{rr'}, \\ \bar{v}^{(r')}(\mathbf{p}, p^a) i\gamma^0 v^{(r)}(\mathbf{p}, p^a) &= \delta_{rr'}, \\ \sum_r u_\alpha^{(r)}(\mathbf{p}, p^a) \bar{u}^{(r)\beta}(\mathbf{p}, p^a) &= \frac{1}{2p^0} d_\alpha^\beta(ip), \\ \sum_r v_\alpha^{(r)}(\mathbf{p}, p^a) \bar{v}^{(r)\beta}(\mathbf{p}, p^a) &= -\frac{1}{2p^0} d_\alpha^\beta(-ip). \end{aligned}$$

Consider a Lorentz transformation matrix  $L(\mathbf{p}, p^a)$  given by

$$L(\mathbf{p}, p^a) = \sqrt{\frac{p^0 + m}{2m}} \mathbb{I} - \frac{1}{\sqrt{2m(p^0 + m)}} \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + p^b \gamma^b). \quad (14)$$

Then we have

$$L^{-1}(\mathbf{p}, p^a) \gamma^\mu L(\mathbf{p}, p^a) = \Lambda_\nu^\mu(\mathbf{p}, p^a) \gamma^\nu, \quad (15)$$

where

$$\begin{aligned} \Lambda_\nu^0(\mathbf{p}, p^a) &= \left( \frac{p^k}{m}, \frac{p^a}{m}, \frac{p^0}{m} \right), \\ \Lambda_\nu^i(\mathbf{p}, p^a) &= \left( g^{ik} + \frac{p^i p^k}{m(p^0 + m)}, \frac{p^i p^a}{m(p^0 + m)}, \frac{p^i}{m} \right), \\ \Lambda_\nu^b(\mathbf{p}, p^a) &= \left( \frac{p^b p^k}{m(p^0 + m)}, g^{ba} + \frac{p^b p^a}{m(p^0 + m)}, \frac{p^b}{m} \right). \end{aligned}$$

The transformation coefficients  $\Lambda_\nu^\mu$  satisfy the relation

$$g_{\mu\nu} \Lambda_\rho^\mu(\mathbf{p}, p^a) \Lambda_\sigma^\nu(\mathbf{p}, p^a) = g_{\rho\sigma},$$

as is expected, and hence they induce the homogenous Lorentz transformation. It follows from equation (15) that

$$L^{-1}(\mathbf{p}, p^a) d(ip) L(\mathbf{p}, p^a) = m(\mathbb{I} + i\gamma^0). \quad (16)$$

The factor  $(\mathbb{I} + i\gamma^0)$  plays a crucial role when constructing wavefunctions, because we find the following key relations from this factor:

$$\begin{aligned} L(\mathbf{p}, p^a) (\mathbb{I} + i\gamma^0) &= \frac{1}{\sqrt{2m(p^0 + m)}} d(ip) (\mathbb{I} + i\gamma^0), \\ (\mathbb{I} + i\gamma^0) L^\dagger(\mathbf{p}, p^a) \eta &= (\mathbb{I} + i\gamma^0) L^{-1}(\mathbf{p}, p^a), \end{aligned}$$

where we have used the relation

$$\gamma^0 L^\dagger(\mathbf{p}, p^a) \gamma^0 = -L^{-1}(\mathbf{p}, p^a).$$

Note that equation (6) leads to the useful relation:

$$\hat{C}L^*(\mathbf{p}, p^a)\hat{C}^{-1} = L(\mathbf{p}, p^a).$$

If we choose the Pauli representation for the gamma matrices, the relation (16) states that the Klein–Gordon divisor is diagonalized by the Lorentz boost (14).

The helicity operator  $h$  is defined in terms of the rank-3 Pauli–Lubanski tensor:

$$h = -\frac{1}{2} \frac{1}{|\mathbf{p}|} w_{045} = \frac{1}{2} \Sigma_k \frac{p^k}{|\mathbf{p}|}, \quad (17)$$

where  $\Sigma_k$  is defined in equation (3), and the complete Pauli–Lubanski tensor is given by

$$w_{\lambda\mu\nu} = \frac{1}{2} \epsilon_{\lambda\mu\nu\rho\alpha\tau} p^\rho \sigma^{\alpha\tau}.$$

By using the representation of gamma matrices given in equation (4), we find that

$$[L(\mathbf{p}, p^a), h] = 0.$$

To diagonalize the helicity operator (17), we introduce the following unitary matrix:

$$S(\mathbf{p}) = \frac{1}{\sqrt{2(1+n^3)}} [(1+n^3)\mathbb{I} + in^2\Sigma_1 - in^1\Sigma_2], \quad (18)$$

where

$$n^k = \frac{p^k}{|\mathbf{p}|}.$$

By using the matrix (18), we can prove the relation

$$S^{-1}(\mathbf{p})hS(\mathbf{p}) = \frac{1}{2}\Sigma_3,$$

where  $\Sigma_3$  is defined in equation (3), hence the helicity operator  $h$  is diagonalized by  $S(\mathbf{p})$ . The definition in equation (18) implies that

$$\hat{C}S^*(\mathbf{p})\hat{C}^{-1} = S(\mathbf{p}).$$

By noting that

$$\hat{C}h^*\hat{C}^{-1} = -h,$$

we find

$$hu^{(r)}(\mathbf{p}, p^a) = \frac{1}{2}\epsilon^{(r)}u^{(r)}(\mathbf{p}, p^a),$$

and

$$hv^{(r)}(\mathbf{p}, p^a) = h\hat{C}u^{(r)*}(\mathbf{p}, p^a) = -\frac{1}{2}\epsilon^{(r)}v^{(r)}(\mathbf{p}, p^a),$$

in the Pauli representation, where  $r$  runs from 1 to 4, and  $\epsilon^{(r)}$  is given by

$$\epsilon^{(r)} = \begin{cases} 1 & \text{for } r = 1, 3, \\ -1 & \text{for } r = 2, 4. \end{cases}$$

Wavefunctions are constructed in the Pauli representation for gamma matrices as follows:

$$\begin{aligned} h_\alpha^\beta u_\beta^{(r)}(\mathbf{p}, p^a) &= \frac{1}{2}\epsilon^{(r)}u_\alpha^{(r)}(\mathbf{p}, p^a), \\ &= \frac{1}{\sqrt{4mp^0}} h_\alpha^\beta [d(ip)L(\mathbf{p}, p^a)S(\mathbf{p})]_\beta^r, \\ &= \sqrt{\frac{m}{p^0}} h_\alpha^\beta \left[ L(\mathbf{p}, p^a) \frac{1}{2} (\mathbb{I} + i\gamma^0) S(\mathbf{p}) \right]_\beta^r, \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{\sqrt{2p^0(p^0+m)}} \left[ (-i\boldsymbol{\gamma} \cdot \mathbf{p} + m) \frac{1}{2} (\mathbb{I} + i\boldsymbol{\gamma}^0) h S(\mathbf{p}) \right]_{\alpha}^r, \\
 &= \frac{1}{2} \frac{1}{\sqrt{2p^0(p^0+m)}} \left[ (-i\boldsymbol{\gamma} \cdot \mathbf{p} + m) \frac{1}{2} (\mathbb{I} + i\boldsymbol{\gamma}^0) S(\mathbf{p}) \Sigma_3 \right]_{\alpha}^r, \tag{19} \\
 \bar{u}^{(r)\beta}(\mathbf{p}, p^a) h_{\beta}^{\alpha} &= \frac{1}{2} \epsilon^{(r)} \bar{u}^{(r)\alpha}(\mathbf{p}, p^a), \\
 &= \sqrt{\frac{m}{p^0}} \left[ S^{-1}(\mathbf{p}) \frac{1}{2} (\mathbb{I} + i\boldsymbol{\gamma}^0) L^{-1}(\mathbf{p}, p^a) \right]_{\beta}^{\alpha} h_{\beta}^{\alpha}, \\
 &= \frac{1}{2} \frac{1}{\sqrt{2p^0(p^0+m)}} \left[ \Sigma_3 S^{-1}(\mathbf{p}) \frac{1}{2} (\mathbb{I} + i\boldsymbol{\gamma}^0) (-i\boldsymbol{\gamma} \cdot \mathbf{p} + m) \right]_{\beta}^{\alpha}, \\
 h_{\alpha}^{\beta} v_{\beta}^{(r)}(\mathbf{p}, p^a) &= -\frac{1}{2} \epsilon^{(r)} v_{\alpha}^{(r)}(\mathbf{p}, p^a), \\
 &= \sqrt{\frac{m}{p^0}} h_{\alpha}^{\beta} \left[ L(\mathbf{p}, p^a) \frac{1}{2} (\mathbb{I} - i\boldsymbol{\gamma}^0) S(\mathbf{p}) \Sigma_3 \hat{C} \right]_{\beta r}^{\alpha}, \\
 &= \frac{1}{2} \frac{1}{\sqrt{2p^0(p^0+m)}} \left[ (i\boldsymbol{\gamma} \cdot \mathbf{p} + m) \frac{1}{2} (\mathbb{I} - i\boldsymbol{\gamma}^0) S(\mathbf{p}) \Sigma_3 \hat{C} \right]_{\alpha r}^{\beta}, \\
 \bar{v}^{(r)\beta}(\mathbf{p}, p^a) h_{\beta}^{\alpha} &= -\frac{1}{2} \epsilon^{(r)} \bar{v}^{(r)\alpha}(\mathbf{p}, p^a), \\
 &= -\sqrt{\frac{m}{p^0}} \left[ \hat{C}^{-1} S^{-1}(\mathbf{p}) \frac{1}{2} (\mathbb{I} - i\boldsymbol{\gamma}^0) L^{-1}(\mathbf{p}, p^a) \right]_{\beta}^{\alpha} h_{\beta}^{\alpha}, \\
 &= -\frac{1}{2} \frac{1}{\sqrt{2p^0(p^0+m)}} \left[ \hat{C}^{-1} \Sigma_3 S^{-1}(\mathbf{p}) \frac{1}{2} (\mathbb{I} - i\boldsymbol{\gamma}^0) (i\boldsymbol{\gamma} \cdot \mathbf{p} + m) \right]_{\beta}^{\alpha}, \tag{20}
 \end{aligned}$$

where the charge conjugation matrix, obtained from equation (6), is given by

$$\hat{C} = \boldsymbol{\gamma}^0 \boldsymbol{\gamma}^4 \boldsymbol{\gamma}^5 \boldsymbol{\gamma}^2 = \begin{pmatrix} \mathbf{0}_{4 \times 4} & -i\sigma_2 & 0 \\ -i\sigma_2 & 0 & -i\sigma_2 \\ 0 & -i\sigma_2 & \mathbf{0}_{4 \times 4} \end{pmatrix} = -\hat{C}^{-1}. \tag{21}$$

To obtain the explicit form of the charge conjugation matrix, we have to fix a representation of the gamma matrices. We thus find the explicit form, equation (21), for the matrix  $\hat{C}$  in the Pauli representation. Explicit forms of wavefunctions that follow from equation (19) to equation (20) are shown in appendix D.1.

#### 4. An eight-dimensional realization of the Clifford algebra in the five-dimensional Galilean covariant spacetime

In this section, we turn to the reduction from the (5 + 1) Minkowski manifold to the (4 + 1) Galilean covariant spacetime. More specifically, we exploit the results found in the previous sections to obtain 8 × 8 gamma matrices (denoted by  $\Gamma$ ) in the Galilean covariant spacetime, from the gamma matrices (denoted by  $\boldsymbol{\gamma}$ ) defined on the extended Minkowski manifold.

Consider the five-dimensional Galilean covariant spacetime with light-cone coordinates,  $x^{\mu}$  ( $\mu = 1, \dots, 5$ ), with the metric tensor:

$$\eta_{\mu\nu} = \begin{pmatrix} \mathbf{1}_{3 \times 3} & \mathbf{0}_{3 \times 2} \\ \mathbf{0}_{2 \times 3} & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}.$$

The coordinate system  $y^\mu$  ( $\mu = 1, 2, 3, 4, 0$ ), defined by (see [2])

$$\mathbf{y} = \mathbf{x}, \quad y^4 = \frac{1}{\sqrt{2}}(x^4 - x^5), \quad y^0 = \frac{1}{\sqrt{2}}(x^4 + x^5), \quad (22)$$

admits the diagonal metric of equation (2). Therefore, the five-dimensional Galilean covariant spacetime corresponds to a (4+1) Minkowski spacetime, so that it is possible to describe non-relativistic theories in a Lorentz-like covariant form. A further reduction, to the Newtonian spacetime, is needed, as explained in [1, 2].

In order to introduce pseudo-tensor interactions of rank 0, 1 and 2 into the five-dimensional Galilean covariant theory, we need a gamma-6 matrix (which corresponds to the gamma-5 matrix in the usual (3+1) Minkowski spacetime) obtained by dimensional reduction from the (5+1) Minkowski spacetime to the (4+1) Minkowski spacetime with light-cone coordinates.

Let  $\Gamma^\mu$  and  $\gamma^\mu$  be  $8 \times 8$  gamma matrices in the five-dimensional Galilean-covariant and Minkowski spacetimes, respectively. They transform as the contravariant vectors in each spacetime. Therefore, we have

$$\begin{aligned} \Gamma &= \gamma, \\ \Gamma^4 &= \frac{1}{\sqrt{2}}(\gamma^4 + \gamma^0), \\ \Gamma^5 &= \frac{1}{\sqrt{2}}(-\gamma^4 + \gamma^0). \end{aligned} \quad (23)$$

The gamma-6 matrix may be taken as

$$\Gamma^6 = \gamma^7,$$

where  $\Gamma^6$  anticommutes with  $\Gamma^\mu$ . Note that neither  $\gamma^1\gamma^2\gamma^3\gamma^4\gamma^0$  nor  $\Gamma^1\Gamma^2\Gamma^3\Gamma^4\Gamma^5$  anticommute with the  $\Gamma^\mu$  s, which satisfy the Clifford algebra:

$$\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}.$$

The parity matrix  $\Pi$  may be expressed by

$$\Pi = \gamma^4\gamma^5\gamma^0,$$

and satisfies the relations

$$\begin{aligned} \Pi\Gamma^k + \Gamma^k\Pi &= 0, & (k = 1, 2, 3), \\ \Pi\Gamma^4 - \Gamma^4\Pi &= 0, \\ \Pi\Gamma^5 - \Gamma^5\Pi &= 0. \end{aligned}$$

These equations are equivalent to imposing the condition given by equation (11).

Since, in the five-dimensional Galilean covariant spacetime, the dimension of algebra is  $2^5 = 32$ , then we take 32 independent gamma matrices given by

$$\Gamma_A = \mathbb{I}, \Gamma^6, \Gamma_\mu, i\Gamma^6\Gamma_\mu, \Sigma_{\mu\nu}, \Gamma^6\Sigma_{\mu\nu},$$

where  $\Sigma_{\mu\nu}$  is defined by

$$\Sigma_{\mu\nu} = \frac{1}{2i}(\Gamma_\mu\Gamma_\nu - \Gamma_\nu\Gamma_\mu). \quad (24)$$

These  $\Gamma$ -matrices satisfy the relation

$$\text{Tr}(\Gamma^A\Gamma_B) = 8\delta_B^A. \quad (25)$$

Since the  $\Gamma^\mu$ s are linear combinations of  $\gamma^\mu$ s and  $\Gamma^6 = \gamma^7$ , we have

$$(\Gamma^\mu)^T = -C^{-1}\Gamma^\mu C = \hat{C}^{-1}(\Gamma^\mu)^\dagger\hat{C}.$$

Thus we find

$$(\Gamma_A C)_{\alpha\beta} = \epsilon_A(\Gamma_A C)_{\beta\alpha}, \quad (26)$$

where

$$\epsilon_A = \begin{cases} +1 & \text{for } C, \Gamma^6\Sigma_{\mu\nu}C, \\ -1 & \text{for } \Gamma^6C, \Gamma_\mu C, i\Gamma^6\Gamma_\mu C, \Sigma_{\mu\nu}C. \end{cases} \quad (27)$$

4.1. The Dirac-type equation in the Pauli representation

In the five-dimensional Galilean covariant spacetime, the Dirac-type equation for massless fields can be cast in the following form:

$$\Lambda(\partial)\psi(x) = 0, \tag{28}$$

with

$$\Lambda(\partial) = -\Gamma^\mu \partial_\mu,$$

where the wavefunction is an eight-component spinor. The adjoint equation to equation (28) is given by

$$\bar{\psi}(x)\Lambda(-\overleftarrow{\partial}) = 0,$$

and

$$\bar{\psi}(x) = \psi^\dagger(x)\eta.$$

Also, we use

$$\eta = i\frac{1}{\sqrt{2}}(\Gamma^4 + \Gamma^5) = i\gamma^0, \tag{29}$$

which agrees with equation (5). Here, we have imposed the relation:

$$[\eta\Lambda(\partial)]^\dagger = \eta\Lambda(-\partial).$$

For the fifth component of the derivative  $\partial_\mu$ , we have the relationship  $\partial_5 = -im$ , which implies the ansatz

$$\psi(x) = e^{-imx^5}\psi(\mathbf{x}, t),$$

or, in the matrix form,

$$\begin{pmatrix} u_1(x) \\ u_2(x) \\ u_3(x) \\ u_4(x) \end{pmatrix} = e^{-imx^5} \begin{pmatrix} u_1(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) \\ u_3(\mathbf{x}, t) \\ u_4(\mathbf{x}, t) \end{pmatrix}$$

where  $u_k(x)$  and  $u_k(\mathbf{x}, t)$  ( $k = 1, 2, 3, 4$ ) are two-component spinors.

The Galilean-covariant  $\Gamma$ -matrices can be expressed in terms of the  $\gamma$ -matrices in the  $(5 + 1)$  Minkowski spacetime. By using equations (8), we obtain the Dirac-type equation in the Pauli representation. If we write it out explicitly, we have

$$\begin{aligned} i\partial_0[u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t)] &= -\frac{1}{2m}\Delta[u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t)], \\ i\partial_0[u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t)] &= -\frac{1}{2m}\Delta[u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t)], \end{aligned} \tag{30}$$

with

$$\begin{aligned} u_1(\mathbf{x}, t) - u_3(\mathbf{x}, t) &= \frac{1}{\sqrt{2}m}\sigma \cdot \nabla[u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t)], \\ u_2(\mathbf{x}, t) + u_4(\mathbf{x}, t) &= \frac{1}{\sqrt{2}m}\sigma \cdot \nabla[u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t)]. \end{aligned} \tag{31}$$

It is convenient to introduce the orthogonal matrix  $R$ :

$$R = \frac{1}{\sqrt{2}} \begin{pmatrix} I & 0 & I & 0 \\ 0 & I & 0 & I \\ I & 0 & -I & 0 \\ 0 & I & 0 & -I \end{pmatrix} = \frac{1}{\sqrt{2}}(\rho_1 + \rho_3). \tag{32}$$

We can utilize this matrix to rotate  $\psi(\mathbf{x}, t)$  in the form

$$\Psi(\mathbf{x}, t) = R\psi(\mathbf{x}, t).$$

Written explicitly in matrix form, it reads

$$\begin{pmatrix} U_1(\mathbf{x}, t) \\ U_2(\mathbf{x}, t) \\ U_3(\mathbf{x}, t) \\ U_4(\mathbf{x}, t) \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} u_1(\mathbf{x}, t) + u_3(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) + u_4(\mathbf{x}, t) \\ u_1(\mathbf{x}, t) - u_3(\mathbf{x}, t) \\ u_2(\mathbf{x}, t) - u_4(\mathbf{x}, t) \end{pmatrix}.$$

Therefore, we obtain from equations (30) to (31) that

$$i\partial_0 U_1(\mathbf{x}, t) = -\frac{1}{2m} \Delta U_1(\mathbf{x}, t), \quad U_2(\mathbf{x}, t) = \frac{1}{\sqrt{2}m} \sigma \cdot \nabla U_1(\mathbf{x}, t), \quad (33)$$

$$U_3(\mathbf{x}, t) = \frac{1}{\sqrt{2}m} \sigma \cdot \nabla U_4(\mathbf{x}, t), \quad i\partial_0 U_4(\mathbf{x}, t) = -\frac{1}{2m} \Delta U_4(\mathbf{x}, t). \quad (34)$$

This result shows that the five-dimensional Galilean-covariant matrices can be obtained by using a similarity transformation which involves the orthogonal matrix  $R$ .

#### 4.2. Explicit forms of the Galilean-covariant gamma matrices

Consider the Dirac Lagrangian, written as

$$\mathcal{L}(x) = \bar{\psi}(x) \Lambda(x) \psi(x),$$

where

$$\Lambda(\partial) = -\Gamma^\mu \partial_\mu.$$

The hermiticity of the Lagrangian leads to the condition given by equation (13). This Lagrangian becomes

$$\mathcal{L}(x) = \bar{\Psi}(x) \tilde{\Lambda}(\partial) \Psi(x), \quad (35)$$

where  $\Psi$  is given by

$$\Psi(x) = e^{-imx^5} \Psi(\mathbf{x}, t) = e^{-imx^5} R\psi(\mathbf{x}, t),$$

and  $\tilde{\Lambda}$  is defined as

$$\tilde{\Lambda}(\partial) = R\Lambda(\partial)R^{-1}. \quad (36)$$

Note that

$$R = R^T = R^{-1}.$$

Therefore, it follows from equation (36) that

$$\tilde{\Gamma}^\mu = R\Gamma^\mu R^{-1}, \quad \tilde{\eta} = R\eta R^{-1}.$$

The Dirac-type equation is obtained from the Lagrangian given by equation (35):

$$\tilde{\Lambda}(\partial)\Psi(\mathbf{x}, t) = 0. \quad (37)$$

If we express the Galilean-covariant gamma matrices in the Pauli representation, then the Dirac-type equation (37) leads to equations (33) to (34).

Explicit forms of the Galilean-covariant gamma matrices are given by using the Pauli representation as follows:

$$\Gamma^k = \begin{pmatrix} \mathbf{0}_{4 \times 4} & 0 & \sigma_k \\ 0 & -\sigma_k & 0 \\ \sigma_k & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}, \quad (k = 1, 2, 3),$$

$$\Gamma^4 = -\sqrt{2}i \begin{pmatrix} \mathbf{0}_{4 \times 4} & 0 & 0 \\ I & 0 & I \\ 0 & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix},$$

$$\Gamma^5 = -\sqrt{2}i \begin{pmatrix} \mathbf{0}_{4 \times 4} & I & 0 \\ 0 & 0 & 0 \\ 0 & I & \mathbf{0}_{4 \times 4} \end{pmatrix},$$

$$\eta = \begin{pmatrix} \mathbf{0}_{4 \times 4} & I & 0 \\ I & 0 & I \\ 0 & I & \mathbf{0}_{4 \times 4} \end{pmatrix} = \rho_1.$$

Moreover, we find

$$\Gamma^6 = \begin{pmatrix} -I & 0 & \mathbf{0}_{4 \times 4} \\ 0 & -I & 0 \\ \mathbf{0}_{4 \times 4} & I & 0 \end{pmatrix} = -\rho_3, \quad \Pi = \begin{pmatrix} \mathbf{0}_{4 \times 4} & 0 & -iI \\ 0 & -iI & 0 \\ iI & 0 & \mathbf{0}_{4 \times 4} \end{pmatrix}.$$

Here we have replaced  $\tilde{\Gamma}^\mu$ ,  $\tilde{\Gamma}^6$ ,  $\tilde{\eta}$  and  $\tilde{\Pi}$  by  $\Gamma^\mu$ ,  $\Gamma^6$ ,  $\eta$  and  $\Pi$ , respectively. It should be noted that the matrix  $\Gamma^6$  is block diagonal.

The chirality operators may be defined as

$$\frac{1}{2}(\mathbb{I} \pm \Gamma^6).$$

They are block diagonal, and the chiral eigenstates are given by

$$\psi_\pm(x) = \frac{1}{2}(\mathbb{I} \pm \Gamma^6)\psi(x),$$

which appear in lower and upper four-component spinors.

### 5. Construction of wavefunctions for the Dirac-type equation

The main advantage of employing a five-dimensional Galilean-covariant theory is that we can perform many calculations in a way analogous to the relativistic treatment. Indeed, many of our non-relativistic equations have the same form as the corresponding equations in relativistic quantum theory, except that they are written in a manifestly covariant form on the (4 + 1) Minkowski spacetime.

Let  $P^\mu$  and  $p^\mu$  be contravariant vectors in the five-dimensional Galilean-covariant and Minkowski spacetimes, respectively. Then they are written as

$$P^\mu = (\mathbf{p}, m, E), \quad p^\mu = (\mathbf{p}, p^4, p^0), \tag{38}$$



with

$$p^0 = \frac{1}{\sqrt{2}}(m + E), \quad p^4 = \frac{1}{\sqrt{2}}(m - E),$$

where we have used equation (22). Moreover, if we impose the conditions

$$P_\mu P^\mu = p_\mu p^\mu = -\kappa_m^2,$$

and

$$\kappa_m = \sqrt{2}m,$$

we find

$$E = \frac{1}{2m} \mathbf{p} \cdot \mathbf{p} + m.$$

We find a similar expression for  $p^0$ :

$$p^0 = \sqrt{\mathbf{p} \cdot \mathbf{p} + (p^4)^2 + \kappa_m^2} = \frac{1}{2\kappa_m} \mathbf{p} \cdot \mathbf{p} + \kappa_m.$$

When we perform the reduction from the six-dimensional to the five-dimensional Minkowski spacetime, the Lorentz boost, equation (14), becomes

$$\begin{aligned} L(\mathbf{p}, p^4) &= \sqrt{\frac{p^0 + \kappa_m}{2\kappa_m}} \mathbb{I} - \frac{1}{\sqrt{2\kappa_m(p^0 + \kappa_m)}} \gamma^0 (\mathbf{p} \cdot \boldsymbol{\gamma} + p^4 \gamma^4) \\ &= L^{-1}(-\mathbf{p}, -p^4). \end{aligned} \tag{39}$$

The Galilean-covariant transformation matrix is obtained by substituting equation (23) into equation (39):

$$\begin{aligned} L(\mathbf{p}, p^4) &= \frac{1}{\sqrt{2\kappa_m(p^0 + \kappa_m)}} \left[ (p^0 + \kappa_m) \mathbb{I} - \frac{1}{\sqrt{2}} (\Gamma^4 + \Gamma^5) \mathbf{p} \cdot \boldsymbol{\Gamma} - \frac{1}{2} (-\Gamma^4 \Gamma^5 + \Gamma^5 \Gamma^4) p^4 \right] \\ &=: G(P). \end{aligned} \tag{40}$$

Hence we find

$$G^{-1}(P) \Gamma^\mu G(P) = Z_\nu^\mu(P) \Gamma^\nu,$$

where

$$\begin{aligned} Z_\nu^i(P) &= \left( \eta^{ik} + \frac{P^i P^k}{m(E + 3m)}, \frac{2P^i}{E + 3m}, \frac{(E + m)P^i}{m(E + 3m)} \right), \\ Z_\nu^4(P) &= \left( \frac{2P^k}{E + 3m}, \frac{4m}{E + 3m}, \frac{E - m}{E + 3m} \right), \\ Z_\nu^5(P) &= \left( \frac{(E + m)P^k}{m(E + 3m)}, \frac{E - m}{E + 3m}, \frac{(E + m)^2}{m(E + 3m)} \right). \end{aligned}$$

The transformation coefficients  $Z_\nu^\mu$  lead to

$$\eta_{\mu\nu} Z_\rho^\mu(P) Z_\sigma^\nu(P) = \eta_{\rho\sigma}.$$

By noting that

$$d(i\mathbf{p}) = -(i\boldsymbol{\gamma} \cdot \mathbf{p} - \kappa_m) = -(i\boldsymbol{\Gamma} \cdot \mathbf{P} - \kappa_m) =: D(iP),$$

we find

$$G^{-1}(P) D(iP) G(P) = \kappa_m \left[ \mathbb{I} + i \frac{1}{\sqrt{2}} (\Gamma^4 + \Gamma^5) \right]. \tag{41}$$

We can prove the following relations:

$$G(P)(\mathbb{I} + \eta) = \frac{1}{\sqrt{2m(E + 3m)}} D(iP)(\mathbb{I} + \eta),$$

$$(\mathbb{I} + \eta)G^\dagger(P)\eta = (\mathbb{I} + \eta)G^{-1}(P),$$

where we have used

$$\eta G^\dagger(P)\eta = G^{-1}(P),$$

with  $\eta$  defined by equation (29). If we choose the Pauli representation for gamma matrices, then equation (41) shows us that the Klein–Gordon divisor in the five-dimensional Galilean covariant spacetime is diagonalized by the Galilean boost, equation (40).

Following the prescription developed in section 3, we can construct wavefunctions for the Dirac-type equation:

$$-(i\gamma \cdot p + \kappa_m)u^{(r)}(\mathbf{p}, p^4) = -(i\Gamma \cdot P + \kappa_m)u^{(r)}(P) = 0,$$

where the matrices  $\Gamma^\mu$  are given by equation (23), in the Pauli representation. The wavefunctions then take the form:

$$h_\alpha^\beta u_\beta^{(r)}(P) = \frac{1}{2} \epsilon^{(r)} u_\alpha^{(r)}(P),$$

$$= \frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[ (-i\Gamma \cdot P + \kappa_m) \frac{1}{2} (\mathbb{I} + \eta) S(\mathbf{P}) \Sigma_3 \right]_\alpha^r, \quad (42)$$

$$\bar{u}^{(r)\beta}(P) h_\beta^\alpha = \frac{1}{2} \epsilon^{(r)} \bar{u}^{(r)\alpha}(P),$$

$$= \frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[ \Sigma_3 S^{-1}(\mathbf{P}) \frac{1}{2} (\mathbb{I} + \eta) (-i\Gamma \cdot P + \kappa_m) \right]_r^\alpha,$$

$$h_\alpha^\beta v_\beta^{(r)}(P) = -\frac{1}{2} \epsilon^{(r)} v_\alpha^{(r)}(P),$$

$$= \frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[ (i\Gamma \cdot P + \kappa_m) \frac{1}{2} (\mathbb{I} - \eta) S(\mathbf{P}) \Sigma_3 \hat{C} \right]_{\alpha r},$$

$$\bar{v}^{(r)\beta}(P) h_\beta^\alpha = -\frac{1}{2} \epsilon^{(r)} \bar{v}^{(r)\alpha}(P),$$

$$= -\frac{1}{2} \frac{1}{\sqrt{(E+m)(E+3m)}} \left[ \hat{C}^{-1} \Sigma_3 S(\mathbf{P}) \frac{1}{2} (\mathbb{I} - \eta) (i\Gamma \cdot P + \kappa_m) \right]^{r\alpha}, \quad (43)$$

where the charge-conjugation matrix  $\hat{C}$  is given by equation (21). These wavefunctions are given explicitly in appendix D.2.

## 6. Concluding remarks

The general idea allowing a covariant treatment of non-relativistic theories is to perform a dimensional reduction from (4 + 1) Minkowski spacetime. However, the  $\gamma^5$ -like matrix has no analogue in odd-dimensional spacetime. Therefore, in this paper, we start with a (5 + 1)-dimensional spacetime.

An eight-dimensional realization of the Clifford algebra in the five-dimensional Galilean covariant spacetime is obtained by reduction from the six-dimensional to the five-dimensional

Minkowski spacetime which encompasses Galilean covariant spacetime. The solutions to the Dirac-type equation in the five-dimensional Galilean covariant spacetime are shown explicitly in the Pauli representation (see appendix D.2). The chiral eigenstates are also obtained by rotating the solution just mentioned above by means of equation (32).

Consider an inverse Galilean transformation, obtained by substituting the direction  $(\mathbf{p}, p^4)$  with  $(-\mathbf{p}, -p^4)$ . Then we can derive the Galilean boost from the Lorentz boost, equation (39),

$$L^{-1}(-\mathbf{p}, -p^4) = L(\mathbf{p}, p^4) = G(P),$$

and hence

$$G(P)\Gamma^\mu G^{-1}(P) = \Gamma'^\nu Z_\nu^\mu(P),$$

where

$$\Gamma'^\nu = (-\Gamma, \Gamma^5, \Gamma^4).$$

It should be mentioned that  $\Gamma^4$  and  $\Gamma^5$  are interchanged by substituting  $p^4$  with  $-p^4$ . Thus, in the massless limit, we find

$$\lim_{m \rightarrow 0} Z_\nu^\mu(P) = \begin{pmatrix} 1 & 0 & 0 & v^1 & 0 \\ 0 & 1 & 0 & v^2 & 0 \\ 0 & 0 & 1 & v^3 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ v^1 & v^2 & v^3 & \frac{1}{2}\mathbf{v} \cdot \mathbf{v} & 1 \end{pmatrix},$$

which is exactly the proper Galilean transformation.

The construction presented in this paper allows a definition of a parity operator as well as a chirality operator. We have included important information related to the Clifford algebra, such as the commutation and anticommutation relations, trace formulae and the Fierz identities, in appendices A, B and C, respectively. A reduction to the  $(4 + 1)$  Minkowski spacetime that encompasses the Galilean covariant spacetime is deduced.

Now the necessary developments to treat Galilean covariant theories for applications to problems like  $\beta$ -decay and to develop a theory like the Galilean covariant version of the Nambu–Jona–Lasinio problem are possible.

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### Appendix A. Commutation and anticommutation relations for the gamma matrices

Hereafter, we provide lists of commutation and anticommutation relations for the gamma matrices. Section A.1 contains these relations for the  $8 \times 8$  representations discussed in section 2.1, for the  $(5 + 1)$  Minkowski spacetime. The corresponding relations for the five-dimensional Galilean covariant spacetime  $\Gamma$ -matrices, introduced in section 4, are given in section A.2.

## A.1. The (5 + 1) Minkowski spacetime

The quantities encountered hereafter have been defined in section 2.1. The matrices  $\Sigma^{[1]}$  are described at the end of the present section:

$$\begin{aligned}
[\gamma^7, \gamma^\mu] &= 2\gamma^7\gamma^\mu, \\
[\gamma^7, i\gamma^7\gamma^\mu] &= 2i\gamma^\mu, \\
[\gamma^7, \sigma^{\mu\nu}] &= 0, \\
[\gamma^7, \gamma^7\sigma^{\mu\nu}] &= 0, \\
[\gamma^7, \sigma^{\lambda\mu\nu}] &= 2\gamma^7\sigma^{\lambda\mu\nu}, \\
[\gamma^\mu, \gamma^\nu] &= 2i\sigma^{\mu\nu}, \\
[\gamma^\mu, i\gamma^7\gamma^\nu] &= -i\gamma^7\{\gamma^\mu, \gamma^\nu\}, \\
[\gamma^\rho, \sigma^{\mu\nu}] &= -2i(g^{\rho\mu}\gamma^\nu - g^{\rho\nu}\gamma^\mu), \\
[\gamma^\lambda, \gamma^7\sigma^{\mu\nu}] &= -\gamma^7\{\gamma^\lambda, \sigma^{\mu\nu}\}, \\
[\gamma^\kappa, \sigma^{\lambda\mu\nu}] &= -\frac{1}{6}\epsilon^{\lambda\mu\nu\rho\sigma\tau}\gamma^7\{\gamma^\kappa, \sigma_{\rho\sigma\tau}\}, \\
[i\gamma^7\gamma^\mu, i\gamma^7\gamma^\nu] &= [\gamma^\mu, \gamma^\nu], \\
[i\gamma^7\gamma^\rho, \sigma^{\mu\nu}] &= i\gamma^7[\gamma^\rho, \sigma^{\mu\nu}], \\
[i\gamma^7\gamma^\lambda, \gamma^7\sigma^{\mu\nu}] &= i\gamma^7\{\gamma^\lambda, \sigma^{\mu\nu}\}, \\
[i\gamma^7\gamma^\rho, \sigma^{\lambda\mu\nu}] &= i\gamma^7\{\gamma^\rho, \sigma^{\lambda\mu\nu}\}, \\
[\sigma^{\mu\nu}, \sigma^{\rho\sigma}] &= 2i(g^{\mu\sigma}\sigma^{\rho\nu} - g^{\mu\rho}\sigma^{\sigma\nu} - g^{\nu\sigma}\sigma^{\rho\mu} + g^{\nu\rho}\sigma^{\sigma\mu}), \\
[\sigma^{\mu\nu}, \gamma^7\sigma^{\rho\sigma}] &= \gamma^7[\sigma^{\mu\nu}, \sigma^{\rho\sigma}], \\
[\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}] &= 2i[(g^{\lambda\rho}\sigma^{\sigma\mu\nu} - g^{\lambda\sigma}\sigma^{\rho\mu\nu}) + (g^{\mu\rho}\sigma^{\sigma\nu\lambda} - g^{\mu\sigma}\sigma^{\rho\nu\lambda}) + (g^{\nu\rho}\sigma^{\sigma\lambda\mu} - g^{\nu\sigma}\sigma^{\rho\lambda\mu})], \\
&= -\frac{1}{6}\epsilon^{\lambda\mu\nu\kappa\tau\eta}[\sigma^{\rho\sigma}, \sigma_{\kappa\tau\eta}]\gamma^7, \\
&= -i\epsilon^{\lambda\mu\nu\kappa\tau\eta}(g_\kappa^\rho\sigma_{\tau\eta}^\sigma - g_\kappa^\sigma\sigma_{\tau\eta}^\rho)\gamma^7, \\
[\gamma^7\sigma^{\mu\nu}, \gamma^7\sigma^{\rho\sigma}] &= [\sigma^{\mu\nu}, \sigma^{\rho\sigma}], \\
[\gamma^7\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}] &= \gamma^7\{\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}\}, \\
[\sigma^{\lambda\mu\nu}, \sigma^{\rho\sigma\tau}] &= 2(i\Sigma^{[\lambda\mu\nu][\rho\sigma\tau]} + \epsilon^{\lambda\mu\nu\rho\sigma\tau}\gamma^7), \\
\{\gamma^7, \gamma^\mu\} &= 0, \\
\{\gamma^7, i\gamma^7\gamma^\mu\} &= 0, \\
\{\gamma^7, \sigma^{\mu\nu}\} &= 2\gamma^7\sigma^{\mu\nu}, \\
\{\gamma^7, \gamma^7\sigma^{\mu\nu}\} &= 2\sigma^{\mu\nu}, \\
\{\gamma^7, \sigma^{\lambda\mu\nu}\} &= 0, \\
\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}, \\
\{\gamma^\mu, i\gamma^7\gamma^\nu\} &= -i\gamma^7[\gamma^\mu, \gamma^\nu], \\
\{\gamma^\lambda, \sigma^{\mu\nu}\} &= -\frac{1}{3}\epsilon^{\lambda\mu\nu\rho\sigma\tau}\sigma_{\rho\sigma\tau}\gamma^7 = 2\sigma^{\lambda\mu\nu}, \\
\{\gamma^\rho, \gamma^7\sigma^{\mu\nu}\} &= -\gamma^7[\gamma^\rho, \sigma^{\mu\nu}], \\
\{\gamma^\rho, \sigma^{\lambda\mu\nu}\} &= 2(g^{\rho\lambda}\sigma^{\mu\nu} + g^{\rho\mu}\sigma^{\nu\lambda} + g^{\rho\nu}\sigma^{\lambda\mu}), \\
\{i\gamma^7\gamma^\mu, i\gamma^7\gamma^\nu\} &= \{\gamma^\mu, \gamma^\nu\}, \\
\{i\gamma^7\gamma^\lambda, \sigma^{\mu\nu}\} &= i\gamma^7\{\gamma^\lambda, \sigma^{\mu\nu}\}, \\
\{i\gamma^7\gamma^\rho, \gamma^7\sigma^{\mu\nu}\} &= -i[\gamma^\rho, \sigma^{\mu\nu}], \\
\{i\gamma^7\gamma^\kappa, \sigma^{\lambda\mu\nu}\} &= i\gamma^7[\gamma^\kappa, \sigma^{\lambda\mu\nu}],
\end{aligned}$$

$$\begin{aligned}
\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\} &= 2(\Sigma^{[\mu\nu][\rho\sigma]} + i\epsilon^{\mu\nu\rho\sigma\eta\xi}\sigma_{\eta\xi}\gamma^7), \\
\{\sigma^{\mu\nu}, \gamma^7\sigma^{\rho\sigma}\} &= \gamma^7\{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\}, \\
\{\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}\} &= 2(\Sigma^{[\rho\sigma][\lambda\mu\nu]} - \epsilon^{\rho\sigma\lambda\mu\nu\eta}\gamma_\eta\gamma^7), \\
\{\gamma^7\sigma^{\mu\nu}, \gamma^7\sigma^{\rho\sigma}\} &= \{\sigma^{\mu\nu}, \sigma^{\rho\sigma}\}, \\
\{\gamma^7\sigma^{\rho\sigma}, \sigma^{\lambda\mu\nu}\} &= \gamma^7[\sigma^{\rho\sigma}\sigma^{\lambda\mu\nu}], \\
\{\sigma^{\lambda\mu\nu}, \sigma^{\rho\sigma\tau}\} &= \frac{1}{3}[2(g^{\lambda\rho}\sigma^{\sigma\tau} + g^{\lambda\sigma}\sigma^{\tau\rho} + g^{\lambda\tau}\sigma^{\rho\sigma})\sigma^{\mu\nu} + 2(g^{\mu\rho}\sigma^{\sigma\tau} + g^{\mu\sigma}\sigma^{\tau\rho} + g^{\mu\tau}\sigma^{\rho\sigma})\sigma^{\nu\lambda} \\
&\quad + 2(g^{\nu\rho}\sigma^{\sigma\tau} + g^{\nu\sigma}\sigma^{\tau\rho} + g^{\nu\tau}\sigma^{\rho\sigma})\sigma^{\lambda\mu}] - \frac{1}{3}i[\gamma^7\gamma^\lambda(\epsilon^{\rho\sigma\tau\mu\xi\xi}\sigma_{\xi\xi}^\nu - \epsilon^{\rho\sigma\tau\nu\xi\xi}\sigma_{\xi\xi}^\mu) \\
&\quad + \gamma^7\gamma^\mu(\epsilon^{\rho\sigma\tau\nu\xi\xi}\sigma_{\xi\xi}^\lambda - \epsilon^{\rho\sigma\tau\lambda\xi\xi}\sigma_{\xi\xi}^\nu) + \gamma^7\gamma^\nu(\epsilon^{\rho\sigma\tau\lambda\xi\xi}\sigma_{\xi\xi}^\mu - \epsilon^{\rho\sigma\tau\mu\xi\xi}\sigma_{\xi\xi}^\lambda)].
\end{aligned}$$

The matrices  $\Sigma^{[\square]}$  are defined by:

$$\begin{aligned}
\Sigma^{[\mu\nu][\rho\sigma]} &= g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\rho\nu}, \\
\Sigma^{[\rho\sigma][\lambda\mu\nu]} &= (g^{\rho\lambda}g^{\sigma\mu} - g^{\sigma\lambda}g^{\rho\mu})\gamma^\nu + (g^{\rho\mu}g^{\sigma\nu} - g^{\sigma\mu}g^{\rho\nu})\gamma^\lambda + (g^{\rho\nu}g^{\sigma\lambda} - g^{\sigma\nu}g^{\rho\lambda})\gamma^\mu, \\
\Sigma^{[\lambda\mu\nu][\rho\sigma\tau]} &= (g^{\lambda\rho}g^{\mu\sigma} - g^{\lambda\sigma}g^{\mu\rho})\sigma^{\nu\tau} + (g^{\lambda\sigma}g^{\mu\tau} - g^{\lambda\tau}g^{\mu\sigma})\sigma^{\nu\rho} + (g^{\lambda\tau}g^{\mu\rho} - g^{\lambda\rho}g^{\mu\tau})\sigma^{\nu\sigma} \\
&\quad + (g^{\mu\rho}g^{\sigma\nu} - g^{\mu\sigma}g^{\nu\rho})\sigma^{\lambda\tau} + (g^{\mu\sigma}g^{\nu\tau} - g^{\mu\tau}g^{\nu\sigma})\sigma^{\lambda\rho} + (g^{\mu\tau}g^{\nu\rho} - g^{\mu\rho}g^{\nu\tau})\sigma^{\lambda\sigma} \\
&\quad + (g^{\nu\rho}g^{\lambda\sigma} - g^{\nu\sigma}g^{\lambda\rho})\sigma^{\mu\tau} + (g^{\nu\sigma}g^{\lambda\tau} - g^{\nu\tau}g^{\lambda\sigma})\sigma^{\mu\rho} + (g^{\nu\tau}g^{\lambda\rho} - g^{\nu\rho}g^{\lambda\tau})\sigma^{\mu\sigma}.
\end{aligned}$$

## A.2. The five-dimensional Galilean covariant spacetime

The quantities described in this appendix are described in section 4. The matrices  $\Sigma^{\mu\nu}$  are defined in equation (24):

$$\begin{aligned}
[\Gamma^6, \Gamma^\mu] &= 2\Gamma^6\Gamma^\mu, \\
[\Gamma^6, i\Gamma^6\Gamma^\mu] &= 2i\Gamma^\mu, \\
[\Gamma^6, \Sigma^{\mu\nu}] &= 0, \\
[\Gamma^6, \Gamma^6\Sigma^{\mu\nu}] &= 0, \\
[\Gamma^\mu, \Gamma^\nu] &= 2i\Sigma^{\mu\nu}, \\
[\Gamma^\mu, i\Gamma^6\Gamma^\nu] &= -i\Gamma^6\{\Gamma^\mu, \Gamma^\nu\}, \\
[\Gamma^\rho, \Sigma^{\mu\nu}] &= -2i(\eta^{\rho\mu}\Gamma^\nu - \eta^{\rho\nu}\Gamma^\mu), \\
[\Gamma^\lambda, \Gamma^6\Sigma^{\mu\nu}] &= -\Gamma^6\{\Gamma^\lambda, \Sigma^{\mu\nu}\}, \\
[i\Gamma^6\Gamma^\mu, i\Gamma^6\Gamma^\nu] &= [\Gamma^\mu, \Gamma^\nu], \\
[i\Gamma^6\Gamma^\rho, \Sigma^{\mu\nu}] &= i\Gamma^6[\Gamma^\rho, \Sigma^{\mu\nu}], \\
[i\Gamma^6\Gamma^\lambda, \Gamma^6\Sigma^{\mu\nu}] &= -i\{\Gamma^\lambda, \Sigma^{\mu\nu}\}, \\
[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}] &= 2i(\eta^{\mu\sigma}\Sigma^{\rho\nu} - \eta^{\mu\rho}\Sigma^{\sigma\nu} - \eta^{\nu\sigma}\Sigma^{\rho\mu} + \eta^{\nu\rho}\Sigma^{\sigma\mu}), \\
[\Sigma^{\mu\nu}, \Gamma^6\Sigma^{\rho\sigma}] &= \Gamma^6[\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}], \\
[\Gamma^6\Sigma^{\mu\nu}, \Gamma^6\Sigma^{\rho\sigma}] &= [\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}], \\
\{\Gamma^6, \Gamma^\mu\} &= 0, \\
\{\Gamma^6, i\Gamma^6\Gamma^\mu\} &= 0, \\
\{\Gamma^6, \Sigma^{\mu\nu}\} &= 2\Gamma^6\Sigma^{\mu\nu}, \\
\{\Gamma^6, \Gamma^6\Sigma^{\mu\nu}\} &= 2\Sigma^{\mu\nu}, \\
\{\Gamma^\mu, \Gamma^\nu\} &= 2\eta^{\mu\nu}, \\
\{\Gamma^\mu, i\Gamma^6\Gamma^\nu\} &= -i\Gamma^6[\Gamma^\mu, \Gamma^\nu], \\
\{\Gamma^\lambda, \Sigma^{\mu\nu}\} &= i(\Gamma^\lambda\Gamma^\nu\Gamma^\mu - \Gamma^\mu\Gamma^\nu\Gamma^\lambda),
\end{aligned}$$

$$\begin{aligned}
\{\Gamma^\rho, \Gamma^6 \Sigma^{\mu\nu}\} &= -\Gamma^6[\Gamma^\rho, \Sigma^{\mu\nu}], \\
\{i\Gamma^6 \Gamma^\mu, i\Gamma^6 \Gamma^\nu\} &= \{\Gamma^\mu, \Gamma^\nu\}, \\
\{i\Gamma^6 \Gamma^\lambda, \Sigma^{\mu\nu}\} &= i\Gamma^6\{\Gamma^\lambda, \Sigma^{\mu\nu}\}, \\
\{i\Gamma^6 \Gamma^\rho, \Gamma^6 \Sigma^{\mu\nu}\} &= -i[\Gamma^\rho, \Sigma^{\mu\nu}], \\
\{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\} &= (\Gamma^\mu \Gamma^\nu \Gamma^\sigma \Gamma^\rho + \Gamma^\rho \Gamma^\sigma \Gamma^\nu \Gamma^\mu) - 2\eta^{\mu\nu} \eta^{\rho\sigma}, \\
\{\Sigma^{\mu\nu}, \Gamma^6 \Sigma^{\rho\sigma}\} &= \Gamma^6\{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\}, \\
\{\Gamma^6 \Sigma^{\mu\nu}, \Gamma^6 \Sigma^{\rho\sigma}\} &= \{\Sigma^{\mu\nu}, \Sigma^{\rho\sigma}\}.
\end{aligned}$$

## Appendix B. Traces of the gamma matrices

In this appendix, we give lists of traces involving the gamma matrices. The  $\gamma$ -matrices defined in section 2.1 for the (5 + 1) Minkowski spacetime are given in appendix B.1, and the  $\Gamma$ -matrices of section 4 for the five-dimensional Galilean covariant spacetime are in appendix B.2.

### B.1. The (5 + 1) Minkowski spacetime

$$\begin{aligned}
\text{Tr}(\gamma_{\mu_1} \cdots \gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\gamma_\mu \gamma_\nu) &= 8g_{\mu\nu}, \\
\text{Tr}(\gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) &= 8(g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma} + g_{\mu\sigma} g_{\nu\rho}), \\
\text{Tr}(\gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) &= 8[(g_{\lambda\mu} g_{\nu\rho} - g_{\lambda\nu} g_{\mu\rho} + g_{\lambda\rho} g_{\mu\nu}) g_{\sigma\tau} - g_{\lambda\rho} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) \\
&\quad - (g_{\lambda\mu} g_{\nu\sigma} - g_{\lambda\nu} g_{\mu\sigma} + g_{\lambda\sigma} g_{\mu\nu}) g_{\tau\rho} - g_{\lambda\sigma} (g_{\mu\tau} g_{\nu\rho} - g_{\mu\rho} g_{\nu\tau}) \\
&\quad + (g_{\lambda\mu} g_{\nu\tau} - g_{\lambda\nu} g_{\mu\tau} + g_{\lambda\tau} g_{\mu\nu}) g_{\rho\sigma} - g_{\lambda\tau} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})], \\
\text{Tr}(\gamma^7) &= 0, \\
\text{Tr}(\gamma^7 \gamma_{\mu_1} \cdots \gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\gamma^7 \gamma_\mu \gamma_\nu) &= 0, \\
\text{Tr}(\gamma^7 \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma) &= 0, \\
\text{Tr}(\gamma^7 \gamma_\lambda \gamma_\mu \gamma_\nu \gamma_\rho \gamma_\sigma \gamma_\tau) &= -8\epsilon_{\lambda\mu\nu\rho\sigma\tau}, \\
\text{Tr}(\sigma_{\mu\nu}) &= 0, \\
\text{Tr}(\sigma_{\lambda\mu\nu}) &= 0, \\
\text{Tr}(\sigma_{\mu\nu} \sigma_{\rho\sigma}) &= 8(g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}), \\
\text{Tr}(\sigma_{\lambda\mu} \sigma_{\nu\rho} \sigma_{\sigma\tau}) &= 8i[(g_{\lambda\tau} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\tau}) g_{\rho\sigma} - (g_{\lambda\sigma} g_{\mu\nu} - g_{\lambda\nu} g_{\mu\sigma}) g_{\tau\rho} \\
&\quad - g_{\lambda\rho} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) + g_{\mu\rho} (g_{\nu\sigma} g_{\lambda\tau} - g_{\nu\tau} g_{\lambda\sigma})], \\
\text{Tr}(\sigma_{\lambda\mu\nu} \sigma_{\rho\sigma\tau}) &= \frac{8}{3}[g_{\lambda\rho} (g_{\mu\sigma} g_{\nu\tau} - g_{\mu\tau} g_{\nu\sigma}) + g_{\lambda\sigma} (g_{\mu\tau} g_{\nu\rho} - g_{\mu\rho} g_{\nu\tau}) + g_{\lambda\tau} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho})], \\
\text{Tr}(\gamma^7 \sigma_{\lambda\mu\nu} \sigma_{\rho\sigma\tau}) &= 8\epsilon_{\lambda\mu\nu\rho\sigma\tau}, \\
\text{Tr}(\gamma^7 \sigma_{\lambda\mu} \sigma_{\nu\rho} \sigma_{\sigma\tau}) &= -8i\epsilon_{\lambda\mu\nu\rho\sigma\tau}.
\end{aligned}$$

### B.2. The five-dimensional Galilean covariant spacetime

$$\begin{aligned}
\text{Tr}(\Gamma_{\mu_1} \cdots \Gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\Gamma_\mu \Gamma_\nu) &= 8\eta_{\mu\nu}, \\
\text{Tr}(\Gamma_\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma) &= 8(\eta_{\mu\nu} \eta_{\rho\sigma} - \eta_{\mu\rho} \eta_{\nu\sigma} + \eta_{\mu\sigma} \eta_{\nu\rho}),
\end{aligned}$$

$$\begin{aligned}
\text{Tr}(\Gamma^6) &= 0, \\
\text{Tr}(\Gamma^6 \Gamma_{\mu_1} \cdots \Gamma_{\mu_n}) &= 0, \quad \text{for } n \text{ odd}, \\
\text{Tr}(\Gamma^6 \Gamma_\mu \Gamma_\nu) &= 0, \\
\text{Tr}(\Gamma^6 \Gamma_\mu \Gamma_\nu \Gamma_\rho \Gamma_\sigma) &= 0, \\
\text{Tr}(\Sigma_{\mu\nu}) &= 0, \\
\text{Tr}(\Sigma_{\mu\nu} \Sigma_{\rho\sigma}) &= 8(\eta_{\mu\rho} \eta_{\nu\sigma} - \eta_{\mu\sigma} \eta_{\nu\rho}).
\end{aligned}$$

### Appendix C. Fierz identities

For later convenience, let us first introduce the following two quantities:

$$(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \frac{1}{4} [(\gamma^A)_{\alpha_1}^{\beta_1} (\gamma^B)_{\alpha_2}^{\beta_2} + (\gamma^A)_{\alpha_1}^{\beta_2} (\gamma^B)_{\alpha_2}^{\beta_1} + (\gamma^B)_{\alpha_1}^{\beta_2} (\gamma^A)_{\alpha_2}^{\beta_1} + (\gamma^B)_{\alpha_1}^{\beta_1} (\gamma^A)_{\alpha_2}^{\beta_2}], \quad (\text{C.1})$$

$$[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \frac{1}{4} [(\gamma^A)_{\alpha_1}^{\beta_1} (\gamma^B)_{\alpha_2}^{\beta_2} - (\gamma^A)_{\alpha_1}^{\beta_2} (\gamma^B)_{\alpha_2}^{\beta_1} - (\gamma^B)_{\alpha_1}^{\beta_2} (\gamma^A)_{\alpha_2}^{\beta_1} + (\gamma^B)_{\alpha_1}^{\beta_1} (\gamma^A)_{\alpha_2}^{\beta_2}]. \quad (\text{C.2})$$

From the definition in equation (C.1), we find the following properties:

$$\begin{aligned}
(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= (\gamma^B, \gamma^A)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\
(\gamma^A + \gamma^B, \gamma^C)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= (\gamma^A, \gamma^C)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} + (\gamma^B, \gamma^C)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\
(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}^{\gamma_1 \gamma_2} (\gamma^C, \gamma^D)_{\gamma_1 \gamma_2}^{\beta_1 \beta_2} &= \frac{1}{2} [(\gamma^A \gamma^C, \gamma^B \gamma^D)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} + (\gamma^A \gamma^D, \gamma^B \gamma^C)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}].
\end{aligned}$$

Similarly, from equation (C.2) we find:

$$\begin{aligned}
[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= [\gamma^B, \gamma^A]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\
[\gamma^A + \gamma^B, \gamma^C]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= [\gamma^A, \gamma^C]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} + [\gamma^B, \gamma^C]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\
[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\gamma_1 \gamma_2} [\gamma^C, \gamma^D]_{\gamma_1 \gamma_2}^{\beta_1 \beta_2} &= \frac{1}{2} ([\gamma^A \gamma^C, \gamma^B \gamma^D]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} - [\gamma^A \gamma^D, \gamma^B \gamma^C]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}).
\end{aligned}$$

Also, the indices admit the following symmetry properties:

$$(\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = (\gamma^A, \gamma^B)_{\alpha_2 \alpha_1}^{\beta_1 \beta_2} = (\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}^{\beta_2 \beta_1}, \quad (\text{C.3})$$

$$[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = -[\gamma^A, \gamma^B]_{\alpha_2 \alpha_1}^{\beta_1 \beta_2} = -[\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_2 \beta_1}. \quad (\text{C.4})$$

The prime Fierz identity comes from the expansion of the identity operator  $I_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}$  in terms of  $(\gamma^A C)_{\alpha_1 \alpha_2}$  and  $(C^{-1} \gamma^B)^{\beta_1 \beta_2}$ :

$$I_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} = \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \sum_{A, B} C_A^B (\gamma^A C)_{\alpha_1 \alpha_2} (C^{-1} \gamma^B)^{\beta_1 \beta_2}. \quad (\text{C.5})$$

The coefficients  $C_A^B$  are determined by using the relation

$$(\gamma^A C)_{\alpha\beta} (C^{-1} \gamma^B)^{\beta\alpha} = \text{Tr}(\gamma^A \gamma^B) = 8\delta_A^B. \quad (\text{C.6})$$

With the operator  $\frac{1}{64} (C^{-1} \gamma_A)^{\alpha_2 \alpha_1} (\gamma^B C)_{\beta_1 \beta_2}$  acting on equation (C.5), we obtain

$$C_A^B = \frac{1}{64} (C^{-1} \gamma_A)^{\alpha_2 \alpha_1} (\gamma^B C)_{\beta_1 \beta_2} \delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \frac{1}{8} \delta_A^B.$$

Hence

$$\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \frac{1}{8} \sum_A (\gamma^A C)_{\alpha_1 \alpha_2} (C^{-1} \gamma_A)^{\beta_2 \beta_1}. \quad (\text{C.7})$$

This is the prime Fierz identity in the (5 + 1) Minkowski spacetime.

Since the relationship (C.6) holds if we replace  $\gamma^A$  with  $\Gamma^A$  [see equation (25)], we can obtain the Fierz identity in the five-dimensional Galilean covariant spacetime:

$$\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} = \frac{1}{8} \sum_A (\Gamma^A C)_{\alpha_1 \alpha_2} (C^{-1} \Gamma_A)^{\beta_2 \beta_1}. \quad (\text{C.8})$$

It should be noted that the Galilean-covariant gamma matrices  $\Gamma^A$  are expressed in terms of  $\gamma^A$  s, so that the relationships (C.1) to (C.4) also hold for  $\Gamma^A$  s.

Recalling equations (9) and (10), we derive from the prime Fierz identity (C.7) that

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} + \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= (\mathbb{I}, \mathbb{I})_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(C)_{\alpha_1 \alpha_2} (C^{-1})^{\beta_2 \beta_1} + \frac{1}{2} (\gamma^7 \sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \gamma^7 \sigma_{\mu\nu})^{\beta_2 \beta_1} \\ &\quad + \frac{1}{8} (\sigma^{\lambda\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \sigma_{\lambda\mu\nu})^{\beta_2 \beta_1}], \end{aligned} \quad (\text{C.9})$$

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= [\mathbb{I}, \mathbb{I}]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(\gamma^7 C)_{\alpha_1 \alpha_2} (C^{-1} \gamma^7)^{\beta_2 \beta_1} + (\gamma^\mu C)_{\alpha_1 \alpha_2} (C^{-1} \gamma_\mu)^{\beta_2 \beta_1} \\ &\quad + (i\gamma^7 \gamma^\mu C)_{\alpha_1 \alpha_2} (C^{-1} i\gamma^7 \gamma_\mu)^{\beta_2 \beta_1} + \frac{1}{2} (\sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \sigma_{\mu\nu})^{\beta_2 \beta_1}]. \end{aligned} \quad (\text{C.10})$$

Similarly, we obtain from equation (C.8), together with equations (26) and (27), that

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} + \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= (\mathbb{I}, \mathbb{I})_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(C)_{\alpha_1 \alpha_2} (C^{-1})^{\beta_2 \beta_1} + \frac{1}{2} (\Gamma^6 \Sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \Gamma^6 \Sigma_{\mu\nu})^{\beta_2 \beta_1}], \end{aligned} \quad (\text{C.11})$$

$$\begin{aligned} \frac{1}{2} (\delta_{\alpha_1}^{\beta_1} \delta_{\alpha_2}^{\beta_2} - \delta_{\alpha_1}^{\beta_2} \delta_{\alpha_2}^{\beta_1}) &= [\mathbb{I}, \mathbb{I}]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2}, \\ &= \frac{1}{8} [(\Gamma^6 C)_{\alpha_1 \alpha_2} (C^{-1} \Gamma^6)^{\beta_2 \beta_1} + (\Gamma^\mu C)_{\alpha_1 \alpha_2} (C^{-1} \Gamma_\mu)^{\beta_2 \beta_1} \\ &\quad + (i\Gamma^6 \Gamma^\mu C)_{\alpha_1 \alpha_2} (C^{-1} i\Gamma^6 \Gamma_\mu)^{\beta_2 \beta_1} + \frac{1}{2} (\Sigma^{\mu\nu} C)_{\alpha_1 \alpha_2} (C^{-1} \Sigma_{\mu\nu})^{\beta_2 \beta_1}]. \end{aligned} \quad (\text{C.12})$$

Further Fierz identities follow from equations (C.9) and (C.10):

$$\begin{aligned} (\gamma^A, \gamma^B)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= \frac{1}{16} [\epsilon_B (\gamma^A \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1})^{\beta_2 \beta_1} \\ &\quad + \frac{1}{32} [\epsilon_B (\gamma^A \gamma^7 \sigma^{\mu\nu} \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B \gamma^7 \sigma^{\mu\nu} \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \gamma^7 \sigma_{\mu\nu})^{\beta_2 \beta_1} \\ &\quad + \frac{1}{96} [\epsilon_B (\gamma^A \sigma^{\lambda\mu\nu} \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B \sigma^{\lambda\mu\nu} \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \sigma_{\lambda\mu\nu})^{\beta_2 \beta_1}, \\ [\gamma^A, \gamma^B]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= \frac{1}{16} [\epsilon_B (\gamma^A \gamma^7 \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B \gamma^7 \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \gamma^7)^{\beta_2 \beta_1} \\ &\quad + \frac{1}{16} [\epsilon_B (\gamma^A \gamma^\mu \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B \gamma^\mu \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \gamma_\mu)^{\beta_2 \beta_1} \\ &\quad + \frac{1}{16} [\epsilon_B (\gamma^A i\gamma^7 \gamma^\mu \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B i\gamma^7 \gamma^\mu \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} i\gamma^7 \gamma_\mu)^{\beta_2 \beta_1} \\ &\quad + \frac{1}{32} [\epsilon_B (\gamma^A \sigma^{\mu\nu} \gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\gamma^B \sigma^{\mu\nu} \gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \sigma_{\mu\nu})^{\beta_2 \beta_1}. \end{aligned}$$

Similarly, from equations (C.11) and (C.12), we have

$$\begin{aligned} (\Gamma^A, \Gamma^B)_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= \frac{1}{16} [\epsilon_B (\Gamma^A \Gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\Gamma^B \Gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1})^{\beta_2 \beta_1} \\ &\quad + \frac{1}{32} [\epsilon_B (\Gamma^A \Gamma^6 \Sigma^{\mu\nu} \Gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\Gamma^B \Gamma^6 \Sigma^{\mu\nu} \Gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \Gamma^6 \Sigma_{\mu\nu})^{\beta_2 \beta_1}, \\ [\Gamma^A, \Gamma^B]_{\alpha_1 \alpha_2}^{\beta_1 \beta_2} &= \frac{1}{16} [\epsilon_B (\Gamma^A \Gamma^6 \Gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\Gamma^B \Gamma^6 \Gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \Gamma^6)^{\beta_2 \beta_1} \\ &\quad + \frac{1}{16} [\epsilon_B (\Gamma^A \Gamma^\mu \Gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\Gamma^B \Gamma^\mu \Gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \Gamma_\mu)^{\beta_2 \beta_1} \\ &\quad + \frac{1}{16} [\epsilon_B (\Gamma^A i\Gamma^6 \Gamma^\mu \Gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\Gamma^B i\Gamma^6 \Gamma^\mu \Gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} i\Gamma^6 \Gamma_\mu)^{\beta_2 \beta_1} \\ &\quad + \frac{1}{32} [\epsilon_B (\Gamma^A \Sigma^{\mu\nu} \Gamma^B C)_{\alpha_1 \alpha_2} + \epsilon_A (\Gamma^B \Sigma^{\mu\nu} \Gamma^A C)_{\alpha_1 \alpha_2}] (C^{-1} \Sigma_{\mu\nu})^{\beta_2 \beta_1}. \end{aligned}$$



### Appendix D. Explicit forms of the wavefunctions

In this appendix, we display the wavefunctions explicitly, in both the (5 + 1) Minkowski spacetime (appendix D.1) and in the 5-dimensional Galilean covariant spacetime (appendix D.2).

#### D.1. The (5 + 1) Minkowski spacetime

The wavefunctions are given by equations (19) to (20). Their explicit expressions are given below:

$$\begin{aligned}
 u^{(1)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 1 \\ 0 \\ \frac{|\mathbf{p}|}{p^0 + m} n^4 \\ \frac{|\mathbf{p}|}{p^0 + m} (i + n^5) \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + in^2 \end{pmatrix}, \\
 \bar{u}^{(1)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1 + n^3)}} (1 + n^3, n^1 - in^2) \\
 &\quad \otimes \sqrt{\frac{p^0 + m}{2p^0}} \left( 1, 0, -\frac{|\mathbf{p}|}{p^0 + m} n^4, -\frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \right), \\
 u^{(2)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 1 \\ 0 \\ \frac{|\mathbf{p}|}{p^0 + m} n^4 \\ \frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1 + n^3 \end{pmatrix}, \\
 \bar{u}^{(2)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1 + n^3)}} (-n^1 - in^2, 1 + n^3) \\
 &\quad \otimes \sqrt{\frac{p^0 + m}{2p^0}} \left( 1, 0, -\frac{|\mathbf{p}|}{p^0 + m} n^4, -\frac{|\mathbf{p}|}{p^0 + m} (i + n^5) \right), \\
 u^{(3)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 0 \\ 1 \\ \frac{|\mathbf{p}|}{p^0 + m} (-i + n^5) \\ -\frac{|\mathbf{p}|}{p^0 + m} n^4 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} 1 + n^3 \\ n^1 + in^2 \end{pmatrix}, \\
 \bar{u}^{(3)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1 + n^3)}} (1 + n^3, n^1 - in^2) \\
 &\quad \otimes \sqrt{\frac{p^0 + m}{2p^0}} \left( 0, 1, -\frac{|\mathbf{p}|}{p^0 + m} (i + n^5), \frac{|\mathbf{p}|}{p^0 + m} n^4 \right), \\
 u^{(4)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0 + m}{2p^0}} \begin{pmatrix} 0 \\ 1 \\ \frac{|\mathbf{p}|}{p^0 + m} (i + n^5) \\ -\frac{|\mathbf{p}|}{p^0 + m} n^4 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1 + n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1 + n^3 \end{pmatrix},
 \end{aligned}$$

$$\begin{aligned} \bar{u}^{(4)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \\ &\quad \otimes \sqrt{\frac{p^0+m}{2p^0}} \left( 0, 1, -\frac{|\mathbf{p}|}{p^0+m}(-i+n^5), \frac{|\mathbf{p}|}{p^0+m}n^4 \right), \\ v^{(1)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0+m}{2p^0}} \begin{pmatrix} \frac{|\mathbf{p}|}{p^0+m}n^4 \\ \frac{|\mathbf{p}|}{p^0+m}(-i+n^5) \\ 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1+n^3 \end{pmatrix}, \\ \bar{v}^{(1)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \\ &\quad \otimes \sqrt{\frac{p^0+m}{2p^0}} \left( \frac{|\mathbf{p}|}{p^0+m}n^4, \frac{|\mathbf{p}|}{p^0+m}(i+n^5), -1, 0 \right), \\ v^{(2)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0+m}{2p^0}} \begin{pmatrix} -\frac{|\mathbf{p}|}{p^0+m}n^4 \\ -\frac{|\mathbf{p}|}{p^0+m}(i+n^5) \\ -1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1 + in^2 \end{pmatrix}, \\ \bar{v}^{(2)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1+n^3)}}(1+n^3, n^1 - in^2) \\ &\quad \otimes \sqrt{\frac{p^0+m}{2p^0}} \left( -\frac{|\mathbf{p}|}{p^0+m}n^4, -\frac{|\mathbf{p}|}{p^0+m}(-i+n^5), 1, 0 \right), \\ v^{(3)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0+m}{2p^0}} \begin{pmatrix} \frac{|\mathbf{p}|}{p^0+m}(i+n^5) \\ -\frac{|\mathbf{p}|}{p^0+m}n^4 \\ 0 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1+n^3 \end{pmatrix}, \\ \bar{v}^{(3)}(\mathbf{p}, p^a) &= \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \\ &\quad \otimes \sqrt{\frac{p^0+m}{2p^0}} \left( \frac{|\mathbf{p}|}{p^0+m}(-i+n^5), -\frac{|\mathbf{p}|}{p^0+m}n^4, 0, -1 \right), \\ v^{(4)}(\mathbf{p}, p^a) &= \sqrt{\frac{p^0+m}{2p^0}} \begin{pmatrix} -\frac{|\mathbf{p}|}{p^0+m}(-i+n^5) \\ \frac{|\mathbf{p}|}{p^0+m}n^4 \\ 0 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1 + in^2 \end{pmatrix}, \end{aligned}$$

$$\bar{v}^{(4)}(\mathbf{p}, p^a) = \frac{1}{\sqrt{2(1+n^3)}}(1+n^3, n^1 - in^2) \otimes \sqrt{\frac{p^0+m}{2p^0}} \left( -\frac{|\mathbf{p}|}{p^0+m}(i+n^5), \frac{|\mathbf{p}|}{p^0+m}n^4, 0, 1 \right),$$

where we have used the notation

$$n^a = \frac{p^a}{|\mathbf{p}|}, \quad (a = 4, 5).$$

### D.2. The five-dimensional Galilean covariant spacetime

In this, we give explicit forms of the wavefunctions given by equations (42) to (43). The symbol  $P$  is a shorthand notation for the five-momentum defined in equation (38):

$$u^{(1)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 1 \\ 0 \\ -\frac{E-m}{E+3m} \\ i2\frac{\sqrt{m(E-m)}}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1+in^2 \end{pmatrix},$$

$$\bar{u}^{(1)}(P) = \frac{1}{\sqrt{2(1+n^3)}}(1+n^3, n^1 - in^2) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left( 1, 0, \frac{E-m}{E+3m}, i2\frac{\sqrt{m(E-m)}}{E+3m} \right),$$

$$u^{(2)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 1 \\ 0 \\ -\frac{E-m}{E+3m} \\ -i2\frac{\sqrt{m(E-m)}}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1+in^2 \\ 1+n^3 \end{pmatrix},$$

$$\bar{u}^{(2)}(P) = \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left( 1, 0, \frac{E-m}{E+3m}, -i2\frac{\sqrt{m(E-m)}}{E+3m} \right),$$

$$u^{(3)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 0 \\ 1 \\ -i2\frac{\sqrt{m(E-m)}}{E+3m} \\ \frac{E-m}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1+in^2 \end{pmatrix},$$

$$\bar{u}^{(3)}(P) = \frac{1}{\sqrt{2(1+n^3)}}(1+n^3, n^1 - in^2) \otimes \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \left( 0, 1, -i2\frac{\sqrt{m(E-m)}}{E+3m}, -\frac{E-m}{E+3m} \right),$$

$$u^{(4)}(P) = \frac{1}{\sqrt{2}} \sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 0 \\ 1 \\ i2\frac{\sqrt{m(E-m)}}{E+3m} \\ \frac{E-m}{E+3m} \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1+in^2 \\ 1+n^3 \end{pmatrix},$$

$$\begin{aligned} \bar{u}^{(4)}(P) &= \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \\ &\quad \otimes \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \left( 0, 1, i2\frac{\sqrt{m(E-m)}}{E+3m}, -\frac{E-m}{E+3m} \right), \\ v^{(1)}(P) &= \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} -\frac{E-m}{E+3m} \\ -2i\frac{\sqrt{m(E-m)}}{E+3m} \\ 1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1+n^3 \end{pmatrix}, \\ \bar{v}^{(1)}(P) &= \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \\ &\quad \otimes \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \left( -\frac{E-m}{E+3m}, 2i\frac{\sqrt{m(E-m)}}{E+3m}, -1, 0 \right), \\ v^{(2)}(P) &= \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} \frac{E-m}{E+3m} \\ -2i\frac{\sqrt{m(E-m)}}{E+3m} \\ -1 \\ 0 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1 + in^2 \end{pmatrix}, \\ \bar{v}^{(2)}(P) &= \frac{1}{\sqrt{2(1+n^3)}}(1+n^3, n^1 - in^2) \\ &\quad \otimes \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \left( \frac{E-m}{E+3m}, 2i\frac{\sqrt{m(E-m)}}{E+3m}, 1, 0 \right), \\ v^{(3)}(P) &= \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 2i\frac{\sqrt{m(E-m)}}{E+3m} \\ \frac{E-m}{E+3m} \\ 0 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} -n^1 + in^2 \\ 1+n^3 \end{pmatrix}, \\ \bar{v}^{(3)}(P) &= \frac{1}{\sqrt{2(1+n^3)}}(-n^1 - in^2, 1+n^3) \\ &\quad \otimes \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \left( -2i\frac{\sqrt{m(E-m)}}{E+3m}, \frac{E-m}{E+3m}, 0, -1 \right), \\ v^{(4)}(P) &= \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \begin{pmatrix} 2i\frac{\sqrt{m(E-m)}}{E+3m} \\ -\frac{E-m}{E+3m} \\ 0 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2(1+n^3)}} \begin{pmatrix} 1+n^3 \\ n^1 + in^2 \end{pmatrix}, \\ \bar{v}^{(4)}(P) &= \frac{1}{\sqrt{2(1+n^3)}}(1+n^3, n^1 - in^2) \\ &\quad \otimes \frac{1}{\sqrt{2}}\sqrt{\frac{E+3m}{E+m}} \left( -2i\frac{\sqrt{m(E-m)}}{E+3m}, -\frac{E-m}{E+3m}, 0, 1 \right). \end{aligned}$$

It is important to remark that the solutions in appendix D have well-defined massless limits.

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